Markovian short rates in multidimensional term structure Lévy models

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We study a bond market model and the related term structure of interest rates in which the prices of zero coupon bonds are driven by a multidimensional Lévy process. We show that the short rate forms a Markov process if and only if the deterministic forward rate volatility coefficients are decomposed into products of two factors where the factor depending on the maturity time is the same for all components. The proof is based on the analysis of sample path properties of the underlying multidimensional process.

1 Introduction

The unifying approach for modeling stochastic bond markets and valuing interest rate derivatives presented by Heath, Jarrow and Morton [10] turned out to be a basic methodology in the last decades. In the proposed term structure model the forward rate was assumed to solve a stochastic differential equation driven by a multidimensional Wiener process. The restrictions imposed on the form of the volatility coefficients lead to specific classes of term structure models including well-known interest rate models (see e.g. Vasiček [20] and Cox, Ingersoll, and Ross [5]). During the nineties, the same framework of term structure models driven by processes with jumps has been investigated in detail. Shirakawa [18] studied a model driven by a Wiener and a Poisson process. Björk, Kabanov, and Runggaldier [1] extended the driving stochastic process to the sum of a diffusion and a marked point process having at most finitely many jumps during every finite time interval. A general jump-diffusion bond market model was introduced and studied by Björk et al. [2].

Among other important issues, the question of the short rate to have the Markov property has attracted a considerable attention in the recent literature on term structures of interest rates. The reason for that fact is the simplification of pricing formulas for bonds and derivatives in the underlying bond market in that case. Carverhill [3] proved that the short rate process is

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Markovian within the Heath-Jarrow-Morton framework with deterministic volatility if and only if the volatility coefficient factorizes into a product of two functions depending only on the actual time and maturity time, respectively. Eberlein and Raible [7] (see also Eberlein [6]) generalised the result of [3] to the case of a model driven by a Lévy process under an additional assumption on the related characteristic function of the marginal distribution which particularly holds for the class of hyperbolic distributions. Küchler and Naumann [13] extended this result to the model containing new examples of driving processes like the class of bilateral gamma processes. This class particularly includes the variance gamma processes which play an important role in recent discussions of stochastic models in financial markets (see, e.g. Madan and Seneta [17] and Madan [16], and more recently, Küchler and Tappe [14] and [15]). Other extensions of the result of [3] were derived in [9] for a model driven by a Wiener and a compound Poisson process with different volatility coefficients, and in [8] for a model driven by a fractional Brownian motion. In the present paper, we investigate a bond market model with deterministic volatility coefficients and the corresponding term structure of interest rates driven by a multidimensional Lévy process. We show that the short rate has a Markov property if and only if the volatility coefficients can be decomposed into the products of two factors depending only on the actual time and maturity time where the maturity time factor is common for all the components of the driving multidimensional process.

The paper is organised as follows. In Section 2, we introduce a bond market model with deterministic volatility coefficients and the associated term structure of interest rates driven by a multidimensional Lévy process. We also derive the relationships between the bond prices, the instantaneous forward rates, as well as the short rate process under a martingale measure. In Section 3, we present the necessary and sufficient conditions on the volatility coefficients under which the short rate forms a Markov process. The current value of the bond price process can then be expressed by means of the current value of the short rate process under this criterion. The proof of this result is based on the analysis of sample path properties of the underlying processes with independent increments and consists of several auxiliary assertions deduced in Section 4.

2 The multidimensional term structure Lévy model

In this section, following the line of the arguments used in [1], we define the basic objects of the bond market model driven by both a multidimensional Lévy process. We refer to [11] and [19; Chapter III, Section 1] for the terminology and notions from stochastic analysis.

Suppose that on some stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T^*]}, Q)$ with a fixed time horizon $T^* > 0$ there exist a multidimensional process $L = (L_t^1, \ldots, L_t^n)_{t \in [0,T^*]}$ the components $L^i = (L_t^i)_{t \in [0,T^*]}$ of which are assumed to be real-valued non-deterministic independent Lévy process with generating triplets $(b_i, c_i, F_i(dy))$, for every $i = 1, \ldots, n$, $n \in \mathbb{N}$. Here, $b_i \in \mathbb{R}$ and $c_i \geq 0$ are some constants, and $F_i(dy)$ is a positive σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying the condition:

$$\int (|y| \wedge 1) \, F_i(dy) < \infty \tag{2.1}$$

for every $i = 1, \ldots, n$, so that we can consider the truncation function $h(y) = 0$, $y \in \mathbb{R}$ (see, e.g. [19; Chapter III, Section 1]). Moreover, we assume that the condition:

$$\int e^{xy} \, \mathbb{1}(|y| > 1) \, F_i(dy) < \infty \tag{2.2}$$

holds, for each $x \in [-M, M]$ and some $M > 0$ fixed, and every $i = 1, \dots, n$, where $\mathbb{1}(\cdot)$ denotes the indication function. The condition of (2.2) guarantees the property that the integrals with respect to the Lévy measures $F_i(dy)$ presented below are well defined, for every $i = 1, \dots, n$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T^*]}$ be the natural filtration of the process L, that is, $\mathcal{F}_t = \sigma(L_s \,|\, 0 \le s \le t)$, for all $t \in [0, T^*]$.

Let us consider a *term structure of bond prices* $\{P(t,T) \,|\, 0 \le t \le T \le T^*\}$, where the (positive) process $P = (P(t,T))_{t \in [0,T]}$ denotes the price of a zero coupon bond at time t maturing at time T which satisfies the normalisation condition:

$$P(T,T) = 1 \tag{2.3}$$

for each $T \in [0, T^*]$. Let us suppose that for all but fixed $T \in [0, T^*]$ the logarithm of the bond price process $P = (P(t,T))_{t \in [0,T]}$ is given by the expression:

$$\ln P(t,T) = \ln P(0,T) + \int_0^t \alpha(s,T) \, ds + \sum_{i=1}^n \int_0^t \sigma_i(s,T) \, dL_s^i \tag{2.4}$$

for all $T \in [0, T^*]$ and $i = 1, \dots, n$. Here, $\sigma_i(t,T)$, $i = 1, \dots, n$, are deterministic positive functions defined on the triangle $\{(t,T) \,|\, 0 \le t \le T \le T^*\}$ which are assumed to be continuously differentiable (so that bounded) in both variables and satisfy the condition:

$$\sigma_i(T,T) = 0 \tag{2.5}$$

for all $T \in [0, T^*]$. Then, the integral with respect to the process L^i in (2.4) is understood in the sense of integration by parts:

$$\int_0^t \sigma_i(s,T) \, dL_s^i = \sigma_i(t,T) \, L_t^i - \int_0^t L_s^i \, d\sigma_i(s,T) \tag{2.6}$$

for all $0 \le t \le T \le T^*$ and every $i = 1, \dots, n$, and the function $\alpha(t,T)$ will be specified below. We will also suppose that we are allowed, by the regularity of the functions, to differentiate under the integral sign, to interchange the order of limits and integrals, as well as to interchange the order of integration and differentiation. We further assume that $|\sigma(t,T)| \le M$, for all $0 \le t \le T \le T^*$, for $M > 0$ fixed above.

Assuming that, for each $t \in [0,T]$ fixed, the bond price $P(t,T)$ is (Q-a.s.) continuously differentiable with respect to the variable T on $[0, T^*]$, let us introduce the corresponding *term structure of interest rates* $\{f(t,T) \,|\, 0 \le t \le T \le T^*\}$, where we have:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \tag{2.7}$$

is the *instantaneous forward rate* contracted at time t for maturity T. On the other hand, integrating the equation in (2.7) and using the condition of (2.3), we get:

$$P(t,T) = \exp\Big(- \int_t^T f(t,u) \, du \Big) \tag{2.8}$$

for all $0 \leq t \leq T \leq T^*$, and hence, we see the one-to-one correspondence between the bond prices and the forward rates. Let us also define the *short rate* process $r = (r(t))_{t \in [0,T^*]}$ by:

$$r(t) = f(t,t) \tag{2.9}$$

being the forward rate at time t for maturity t, and the associated with it *money account* process $B = (B(t))_{t \in [0,T^*]}$ by:

$$B(t) = \exp\left(\int_0^t r(s)\, ds \right) \tag{2.10}$$

playing the role of a *numéraire* in the model. Then, setting:

$$\alpha(t,T) = r(t) - \sum_{i=1}^n \theta_i\big(\sigma_i(t,T)\big) \tag{2.11}$$

for all $t \in [0,T]$, by means of the arguments in [11; Chapter II, Section 2], we conclude that the *discounted bond price* process $(P(t,T)/B(t))_{t \in [0,T]}$ forms an $(\mathbb{F}, Q)$-martingale (see [2; Section 5]). Here, $\theta_i(x)$ is a cumulant function of the process L^i defined by:

$$\theta_i(x) = b_i\, x + \frac{c_i x^2}{2} + \int \Big(e^{xy} - 1 \Big)\, F_i(dy) \tag{2.12}$$

for all $x \in [-M, M]$ and every $i = 1, \dots, n$.

Hence, using the expression (2.11) for $\alpha(t,T)$, we get that, under the measure Q and for each $T \in [0,T^*]$, the logarithm of the bond price in (2.4) admits the representation:

$$\ln P(t,T) = \ln P(0,T) + \int_0^t r(s)\, ds + \sum_{i=1}^n \int_0^t \sigma_i(s,T)\, dL_s^i - \sum_{i=1}^n \int_0^t \theta_i\big(\sigma_i(s,T)\big)\, ds \tag{2.13}$$

and the forward rate process in (2.7) takes the form:

$$f(t,T) = f(0,T) - \sum_{i=1}^n \int_0^t \gamma_i(s,T)\, dL_s^i + \sum_{i=1}^n \int_0^t \theta_i'\big(\sigma_i(s,T)\big)\, \gamma_i(s,T)\, ds \tag{2.14}$$

for all $t \in [0,T]$. Here, we define $\gamma_i(t,T) = \partial\sigma_i(t,T)/(\partial T)$, for all $t \in [0,T]$ and every $i = 1, \dots, n$. Therefore, the short rate process in (2.9) is given by:

$$r(t) = f(0,t) - Z(t) + \sum_{i=1}^n \int_0^t \theta_i'\big(\sigma_i(s,t)\big)\, \gamma_i(s,t)\, ds \tag{2.15}$$

where we set:

$$Z(t) = \sum_{i=1}^n \int_0^t \gamma_i(s,t)\, dL_s^i \tag{2.16}$$

for all $t \in [0,T^*]$.

Note that, in the Heath-Jarrow-Morton approach (see [10]) one starts with the specification of the forward rates (2.14), so that the discounted bond prices turn out to be $(\mathbb{F}, Q)$-martingales, or, in other words, Q is a *martingale measure*. In this case, integrating the expression in (2.14), we easily get the following representation for the bond price in (2.8):

$$
\begin{aligned}
&P(t,T) && (2.17)\\
&= \frac{P(0,T)}{P(0,t)} \exp\Big(\sum_{i=1}^n \int_t^T \int_0^t \gamma_i(s,u)\, dL_s^i\, du - \sum_{i=1}^n \int_t^T \int_0^t \theta_i'\big(\sigma_i(s,u)\big)\gamma_i(s,u)\, ds\, du \Big)
\end{aligned}
$$

where, by virtue of the conditions of (2.5), we have:

$$
\sigma_i(t,T) = \int_t^T \gamma_i(t,v)\, dv \tag{2.18}
$$

for all $0 \le t \le T \le T^*$ and every $i = 1, \ldots, n$. Since the process $(P(t,T)/B(t))_{t\in[0,T]}$ turns out to be an $(\mathbb{F}, Q)$-martingale under the condition of (2.11), according to the expression in (2.10), it is easily shown that the bond price $P = (P(t,T))_{t\in[0,T]}$ can be represented as:

$$
P(t,T) = E\Big[\exp\Big(- \int_t^T r(s)\, ds \Big) \Big| \mathcal{F}_t \Big] \tag{2.19}
$$

for all $t \in [0,T]$. Observe that, when the short rate $r = (r(t))_{t\in[0,T^*]}$ is an $(\mathbb{F}, Q)$-Markov process, the expression in (2.19) takes the form:

$$
P(t,T) = E\Big[\exp\Big(- \int_t^T r(s)\, ds \Big) \Big| r(t) \Big] \tag{2.20}
$$

so that $P(t,T) = H(t, r(t), T)$, for all $0 \le t \le T \le T^*$, and some Borel function H defined on $[0,T] \times \mathbb{R} \times [0,T^*]$. In this case, we see from the expression in (2.20) that the current value of the bond price can be expressed by means of the current value of the short rate. We finally note from the expression in (2.15) that the short rate $r = (r(t))_{t\in[0,T^*]}$ is a Markov process if and only if so is the process $Z = (Z(t))_{t\in[0,T^*]}$ defined in (2.16).

3 The results

In this section, we formulate and prove the following main result of the paper which extends the results of [3], [7], [13], and [9] to the case of a multidimensional Lévy term structure model.

Theorem 3.1. *Let $L^i = (L_t^i)_{t\in[0,T^*]}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, be real-valued nondeterministic independent Lévy processes with triplets $(b_i, c_i, F_i(dy))$ satisfying the conditions of (2.1) and (2.2). Suppose that the both functions $t \mapsto \gamma_i(t,T)$ and $t \mapsto \gamma_i(t,T^*)$, for every $i = 1, \ldots, n$, are nonconstant on $[0,T]$, for each $T \in [0,T^*]$. Then, the short rate process $r = (r(t))_{t\in[0,T^*]}$ is Markovian if and only if there exist continuously differentiable functions $\eta_i(t)$, $t \in [0,T^*]$, for $i = 1, \ldots, n$, and a function $\zeta(T) > 0$, for $T \in [0,T^*]$, not depending on $i = 1, \ldots, n$, such that the equality:*

$$
\gamma_i(t,T) = \eta_i(t)\, \zeta(T) \tag{3.1}
$$

holds, for all $0 \le t \le T \le T^*$ and every $i = 1, \ldots, n$.

The proof of this assertion is based on several auxiliary technical lemmata which are deduced in Section 4 below. We further assume that $L^i = (L^i_t)_{t\in[0,T^*]}$, $i = 1, \ldots, n$, are real-valued nondeterministic independent Lévy processes. We start with a simple extension of an assertion from [7] to the case of a multidimensional Lévy process which is proved in Section 4 below.

Lemma 3.2. *Suppose that $Z = (Z(t))_{t\in[0,T^*]}$ from (2.16) is an $(\mathbb{F}, Q)$-Markov process. Then, for each $0 \le T \le S \le T^*$ fixed, the expression:*

$$\sum_{i=1}^{n} \int_0^T \gamma_i(t, S)\, dL^i_t = G_n\bigg(\sum_{i=1}^{n} \int_0^T \gamma_i(t, T)\, dL^i_t\bigg) \quad (Q\text{-}a.s.) \tag{3.2}$$

holds, with a Borel function G_n, $n \in \mathbb{N}$.

The proof of the next assertion is given in [13; Lemma 3.1] and is also presented in the next section for completeness.

Lemma 3.3. *Let $f_i(t)$ and $g_i(t)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, be continuously differentiable nonconstant functions on $[0, T]$, for some $T \in (0, T^*]$ fixed. Suppose that $f_i(t)$ and $g_i(t)$ are affine independent on $[0, T]$, that is, there are no constants $a_i, h_i \in \mathbb{R}$ such that $f_i(t) = a_i g_i(t) + h_i$, for all $t \in [0, T]$ and every $i = 1, \ldots, n$. Then, the distribution of the vector:*

$$\bigg(\int_0^T f_i(t)\, dL^i_t, \int_0^T g_i(t)\, dL^i_t\bigg) \tag{3.3}$$

has a nonzero absolutely continuous part (with respect to the Lebesgue measure λ_2 on $\mathbb{R}^2$), for every $i = 1, \ldots, n$.

The proof of this result is based on the following assertion which is proved in [12; Theorem 3.1].

Lemma 3.4. *Let $f_i(t)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, be continuously differentiable functions on $[0, T]$, for some $T \in (0, T^*]$ fixed. Suppose that the equality:*

$$H_{n,i}\bigg(\int_0^T f_i(t)\, dL^i_t\bigg) = L^i_T \quad (Q\text{-}a.s.) \tag{3.4}$$

holds, with a Borel function $H_{n,i}$, for every $i = 1, \ldots, n$. Then, the function $f_i(t)$ is necessarily a constant on $[0, T]$, for every $i = 1, \ldots, n$.

As a next step we shall prove in Section 4 below an assertion being an extension of corresponding results from [7], [12], [13], and [9] to the case of a multidimensional Lévy process.

Lemma 3.5. *Let $f_i(t)$ and $g_i(t)$, $i = 1, \ldots, n$, be continuously differentiable nonconstant functions on $[0, T]$, for some $T \in (0, T^*]$ fixed. Suppose that the equality:*

$$\sum_{i=1}^{n} \int_0^T f_i(t)\, dL^i_t = G_n\bigg(\sum_{i=1}^{n} \int_0^T g_i(t)\, dL^i_t\bigg) \quad (Q\text{-}a.s.) \tag{3.5}$$

holds, with a Borel function G_n, $n \in \mathbb{N}$. Then, there exists a constant a, not depending on $i = 1, \ldots, n$, such that $f_i(t) = a g_i(t)$, for all $t \in [0, T]$ and every $i = 1, \ldots, n$.

The proof of this result is given in Section 4 below as well. We continue with the proof of the main result stated above.

Proof of Theorem 3.1. Let us first suppose that the functions $\gamma_i(t,T)$, $0 \le t \le T \le T^*$, $i = 1, \dots, n$, satisfy the conditions of (3.1). In this case, the process $Z = (Z(t))_{t \in [0,T^*]}$ defined in (2.16) can be represented in the form:

$$Z(t) = \zeta(t) \left(\sum_{i=1}^n \int_0^t \eta_i(s) \, dL_s^i \right) \tag{3.6}$$

for all $t \in [0, T^*]$. Therefore, we may conclude that Z is a Markov process and so is the process $r = (r(t))_{t \in [0,T^*]}$ from (2.15).

We now assume that $r = (r(t))_{t \in [0,T^*]}$ is a Markov process, and thus, by virtue of the expression in (2.15) and (2.16), the Markov property holds for the process $Z = (Z(t))_{t \in [0,T^*]}$ as well. Moreover, it follows from the assertion of Lemma 3.2 that, for each $0 \le T \le S \le T^*$ fixed, the expression in (3.2) is satisfied, for a Borel function G_n, $n \in \mathbb{N}$. Hence, by applying the assertion of Lemma 3.5 to the functions $t \mapsto \gamma_i(t, T^*)$ and $t \mapsto \gamma_i(t,T)$, $i = 1, \dots, n$, we get that the decomposition:

$$\gamma_i(t, T^*) = \xi(T, T^*) \, \gamma_i(t,T) \tag{3.7}$$

holds, with some function $\xi(T, T^*)$, not depending on t, for all $0 \le t \le T \le T^*$ and every $i = 1, \dots, n$. Observe that, since the functions $t \mapsto \gamma_i(t,T)$ are assumed to be nonconstant on $[0,T]$, we may conclude from the decomposition in (3.7) that $\xi(T, T^*) \neq 0$, for each $T \in (0, T^*]$ fixed. Recall that, since the functions $\sigma_i(t,T)$, $i = 1, \dots, n$, are assumed to be continuously differentiable on the triangle $\{(t,T) \,|\, 0 \le t \le T \le T^*\}$, the functions $T \mapsto \gamma_i(t,T)$, $i = 1, \dots, n$, are continuous on $\{(t,T) \,|\, 0 \le t \le T \le T^*\}$. Then, we have even $\xi(T, T^*) > 0$ in (3.7), because of the obvious property $\xi(T^*, T^*) = 1$ and a continuity argument. Otherwise, it would follow that $\gamma_i(t,T) = 0$, for all $t \in [0,T]$ and some $T \in (0, T^*]$ fixed, for every $i = 1, \dots, n$, that is excluded by assumption. Thus, defining $\eta_i(t) = \gamma_i(t, T^*)$, $i = 1, \dots, n$, and $\zeta(T) = 1/\xi(T, T^*) > 0$, for each $0 \le t \le T \le T^*$, we obtain the decompositions in (3.1). The continuous differentiability of the functions $\eta_i(t)$, $t \in [0, T^*]$, $i = 1, \dots, n$, follows directly from the assumption on the functions $t \mapsto \gamma_i(t, T^*)$ to be continuously differentiable on $[0, T^*]$, respectively. $\square$

Remark 3.6. In the assumptions of Theorem 3.1, we observe from the expressions in (2.18) and (3.1) that the forward rate process in (2.14) admits the representation:

$$\begin{aligned} f(t,T) = {} & f(0,T) - \frac{\zeta(T)}{\zeta(t)} Z(t) \\ & + \zeta(T) \sum_{i=1}^n \int_0^t \left(b_i \, \eta_i(s) + c_i \, \eta_i^2(s) \int_s^T \zeta(u) \, du + \int e^{\eta_i(s) \int_s^T \zeta(u) \, du \, y} \eta_i(s) \, y \, F_i(dy) \, ds \right) \end{aligned} \tag{3.8}$$

where the process $Z = (Z(t))_{t \in [0,T^*]}$ is given by (3.6) and the short rate process from (2.15) takes the form:

$$\begin{aligned} r(t) = {} & f(0,t) - Z(t) \\ & + \zeta(t) \sum_{i=1}^n \int_0^t \left(b_i \, \eta_i(s) + c_i \, \eta_i^2(s) \int_s^t \zeta(u) \, du + \int e^{\eta_i(s) \int_s^t \zeta(u) \, du \, y} \eta_i(s) \, y \, F_i(dy) \, ds \right) \end{aligned} \tag{3.9}$$

for all $t \in [0, T^*]$.

Example 3.7. Suppose that $L^i = (L^i_t)_{t\in[0,T^*]}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, are bilateral gamma processes, that is, they are Lévy processes with the triplets $(0, 0, F_i(dy))$, where we have:

$$F_i(dy) = \left(\frac{\alpha_{i,+}}{y} \, e^{-\lambda_{i,+} y} \, \mathbb{1}(y > 0) + \frac{\alpha_{i,-}}{-y} \, e^{\lambda_{i,-} y} \, \mathbb{1}(y < 0) \right) dy \tag{3.10}$$

and $\lambda_{i,+}, \lambda_{i,-}, \alpha_{i,+}, \alpha_{i,-}$, $i = 1, \ldots, n$, are some positive parameters (see e.g. [13; Section 5]). In this case, if the condition:

$$\big|\sigma(t,T)\big| = \left|\eta_i(t) \int_t^T \zeta(u) du\right| < \min\big\{\lambda_{i,+}, \lambda_{i,-}\big\} \tag{3.11}$$

holds, for all $0 \le t \le T \le T^*$ and $x \in \mathbb{R}$, and every $i = 1, \ldots, n$, then the conditions in (2.2) are satisfied with some $M < \min\{\lambda_{i,+}, \lambda_{i,-}\}$. Thus, the assertion of Theorem 3.1 holds and the expressions in (3.8) and (3.9) take the explicit form:

$$\begin{aligned} f(t,T) &= f(0,T) - \frac{\zeta(T)}{\zeta(t)} \, Z(t) \\ &+ \sum_{i=1}^n \int_0^t \left(\frac{\alpha_{i,+} \eta_i(s) \zeta(T)}{\lambda_{i,+} - \eta_i(s) \int_s^T \zeta(u) du} - \frac{\alpha_{i,-} \eta_i(s) \zeta(T)}{\lambda_{i,-} + \eta_i(s) \int_s^T \zeta(u) du} \right) ds \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} r(t) &= f(0,t) - Z(t) \\ &+ \sum_{i=1}^n \int_0^t \left(\frac{\alpha_{i,+} \eta_i(s) \zeta(t)}{\lambda_{i,+} - \eta_i(s) \int_s^t \zeta(u) du} - \frac{\alpha_{i,-} \eta_i(s) \zeta(t)}{\lambda_{i,-} + \eta_i(s) \int_s^t \zeta(u) du} \right) ds \end{aligned} \tag{3.13}$$

where the process $Z = (Z(t))_{t\in[0,T^*]}$ is given by (3.6).

4 The proofs

In this section, we present the proof of the auxiliary assertions formulated in Section 3.

We start with the proof of a simple extension of an assertion from [7] to the case of a driving multidimensional Lévy process.

Proof of Lemma 3.2. Observe that if the process $Z = (Z(t))_{t\in[0,T^*]}$ from (2.16) is Markovian, then we have:

$$E\big[Z(S) \,\big|\, \mathcal{F}_T\big] = E\big[Z(S) \,\big|\, Z(T)\big] \quad (Q\text{-a.s.}) \tag{4.1}$$

for all $0 \le T \le S \le T^*$. Since the integrands $\gamma(t,T)$ for $0 \le t \le T \le T^*$ are deterministic functions, it follows from the independence of increments of the processes $L = (L_t)_{t\in[0,T^*]}$ that:

$$E\big[Z(S) \,\big|\, \mathcal{F}_T\big] = E\bigg[\sum_{i=1}^n \int_0^T \gamma_i(t,S) \, dL^i_t \,\bigg|\, \mathcal{F}_T\bigg] + E\bigg[\sum_{i=1}^n \int_T^S \gamma_i(t,S) \, dL^i_t\bigg] \quad (Q\text{-a.s.}) \tag{4.2}$$

and

$$E\big[Z(S)\,\big|\,Z(T)\big] = E\bigg[\sum_{i=1}^{n}\int_0^T \gamma_i(t,S)\,dL_t^i\,\bigg|\,Z(T)\bigg] + E\bigg[\sum_{i=1}^{n}\int_T^S \gamma_i(t,S)\,dL_t^i\bigg] \quad (Q\text{-a.s.}) \tag{4.3}$$

for all $0 \le T \le S \le T^*$. Hence, getting the expressions in (4.1)-(4.3) together, we obtain:

$$\sum_{i=1}^{n}\int_0^T \gamma_i(t,S)\,dL_t^i = E\bigg[\sum_{i=1}^{n}\int_0^T \gamma_i(t,S)\,dL_t^i\,\bigg|\,\sum_{i=1}^{n}\int_0^T \gamma_i(t,T)\,dL_t^i\bigg] \quad (Q\text{-a.s.}) \tag{4.4}$$

that immediately implies the desired assertion. $\square$

Proof of Lemma 3.3. (i) In order to prove the desired assertion, let us first assume that the process L^i has the triplet $(b_i, c_i, 0)$ with some $c_i > 0$, for any $i = 1, \dots, n$. In this case, we have $L_t^i = b_i t + W_t^i$, for $t \in [0, T^*]$, where $W^i = (W_t^i)_{t \in [0,T^*]}$ denotes the continuous martingale part of the process L^i, for every $i = 1, \dots, n$. Then, the process W^i is Gaussian, and thus, by means of Itô's isometry, we get:

$$\begin{aligned} E\bigg[\bigg(\int_0^T \big(f_i(t) - a_i\,g_i(t)\big)\,dL_t^i\bigg)^2\bigg] &= \int_0^T \big(f_i(t) - a_i\,g_i(t)\big)^2\,d\langle L^i\rangle_t \\ &= \int_0^T f_i^2(t)\,d\langle L^i\rangle_t - 2a_i \int_0^T f_i(t)\,g_i(t)\,d\langle L^i\rangle_t + a_i^2 \int_0^T g_i^2(t)\,d\langle L^i\rangle_t \end{aligned} \tag{4.5}$$

which is strictly positive, for all $a_i \in \mathbb{R}$, by assumption. On the other hand, it is seen that the expression in the right-hand side of (4.5) represents a quadratic polynomial in $a_i \in \mathbb{R}$, and that expression is strictly positive obviously if and only if the inequality:

$$\int_0^T f_i(t)\,g_i(t)\,d\langle L^i\rangle_t < \int_0^T f_i^2(t)\,d\langle L^i\rangle_t \int_0^T g_i^2(t)\,d\langle L^i\rangle_t \tag{4.6}$$

holds. However, the latter fact means that the distribution of the vector in (3.3) have a nonzero absolutely continuous part (with respect to the Lebesgue measure on $\mathbb{R}^2$). Note that, since any process L^i with the triplet $(b_i, c_i, F_i(dy))$ can be decomposed as $L_t^i = b_i t + W_t^i + J_t^i$, for $t \in [0, T^*]$, where the continuous martingale part W^i is independent of the pure jump part $J^i = (J_t^i)_{t \in [0,T^*]}$ which has a triplet $(0, 0, F_i(dy))$, it remains us to prove the desired assertion for pure jump processes L^i, $i = 1, \dots, n$, only.

(ii) Let us now assume that L^i has the triplet $(0, 0, \lambda_i \mathbb{1}(y = 1))$ with some $\lambda_i > 0$ fixed, for any $i = 1, \dots, n$. In this case, L^i is a Poisson process of intensity λ_i and denote by $(\tau_m^i)_{m \in \mathbb{N}}$ the sequence of its jump times, for any $i = 1, \dots, n$. Define the mapping $\phi_i(u, v)$ by $\phi_i(u, v) = (f_i(u) + f_i(v), g_i(u) + g_i(v))$, for $(u, v) \in [0, T]^2$. Then, by virtue of the assumptions on the functions $f_i(t)$ and $g_i(t)$, the mapping $\phi_i(u, v)$ is continuously differentiable with a nonzero Jacobian determinant:

$$D_i(u, v) = \det \begin{pmatrix} f_i'(u) & f_i'(v) \\ g_i'(u) & g_i'(v) \end{pmatrix} \tag{4.7}$$

at least in an open neighbourhood $U_i(u_0, v_0)$ of some point $(u_0, v_0) \in (0, t)^2$ fixed. In this case, $\phi_i(u, v)$ maps $U_i(u_0, v_0)$ bijectively to an open neighborhood $V_i(\phi_i(u_0, v_0))$ of the point

$\phi_i(u_0,v_0) = (f_i(u_0)+f_i(v_0), g_i(u_0)+g_i(v_0))$, and that the inverse mapping $\phi_i^{-1}(x,y)$ is continuously differentiable. Note that, because of the symmetry of the mapping $\phi_i(u,v)=\phi_i(v,u)$, there is no restriction to assume that $U_i(u_0,v_0)$ is symmetric, so that $(u,v)\in U_i(u_0,v_0)$ if and only if $(v,u)\in U_i(u_0,v_0)$. In particular, the set $U_i(u_0,v_0)\cap\Delta_T$ has a positive Lebesgue measure, where we set $\Delta_T = \{(u,v)\in\mathbb{R}^2 \,|\, 0\le u\le v\le T\}$, for some $i=1,\dots,n$.

It follows from the fact that L^i is a Poisson process that $P(\tau_2^i\le T<\tau_3^i)>0$, for any $i=1,\dots,n$. Moreover, it follows from the properties of the Poisson process that the couple (τ_1^i,τ_2^i) has a strictly positive density $h_i(u,v)$ on Δ_T, and it is uniformly distributed under $\{\tau_2^i\le T<\tau_3^i\}$. Then, we have $Q((\tau_1^i,\tau_2^i)\in U_i(u_0,v_0)\,|\,\tau_2^i\le T<\tau_3^i)>0$, so that $Q((\tau_1^i,\tau_2^i)\in U_i(u_0,v_0),\tau_2^i\le T<\tau_3^i)>0$ holds. Hence, for any Borel set $A\in\mathcal{B}(\mathbb{R}^2)$, we have:

$$
\begin{aligned}
&Q\big(\phi_i(\tau_1^i,\tau_2^i)\in A \,\big|\, (\tau_1^i,\tau_2^i)\in U_i(u_0,v_0),\tau_2^i\le T<\tau_3^i\big)\,Q\big((\tau_1^i,\tau_2^i)\in U_i(u_0,v_0)\,\big|\,\tau_2^i\le T<\tau_3^i\big) \qquad (4.8)\\
&= Q\big(\phi_i(\tau_1^i,\tau_2^i)\in A, (\tau_1^i,\tau_2^i)\in U_i(u_0,v_0)\,\big|\,\tau_2\le T<\tau_3\big)\\
&= \int_{\phi_i^{-1}(A)\cap U_i(u_0,v_0)} h_i(u,v)\,dudv = \int_{\phi^{-1}(A)} h_i(u,v)\,\mathbb{1}\big((u,v)\in U_i(u_0,v_0)\big)\,dudv\\
&= \int_A h_i\big(\phi_i^{-1}(x,y)\big)\,D_i^{-1}\big(\phi_i^{-1}(x,y)\big)\,\mathbb{1}\big((x,y)\in V_i(\phi(u_0,v_0))\big)\,dxdy
\end{aligned}
$$

where we mean $D^{-1}(u,v)=1/D(u,v)$, for $(u,v)\in[0,T]$. Hence, we may conclude that the distribution of the vector $\phi_i(\tau_1^i,\tau_2^i)$ has an absolutely continuous part. Therefore, recalling the fact that:

$$
\left(\int_0^T f_i(t)\,dL_t^i, \int_0^T g_i(t)\,dL_t^i\right) = \phi_i(\tau_1^i,\tau_2^i) \quad \text{on} \quad \big\{(\tau_1^i,\tau_2^i)\in U_i(u_0,v_0),\tau_2^i\le T<\tau_3^i\big\} \qquad (4.9)
$$

we see that the distribution given by:

$$
Q\big(\phi_i(\tau_1^i,\tau_2^i)\in A \,\big|\, (\tau_1^i,\tau_2^i)\in U_i(u_0,v_0),\tau_2^i\le T<\tau_3^i\big)\,Q\big((\tau_1^i,\tau_2^i)\in U_i(u_0,v_0),\tau_2^i\le T<\tau_3^i\big) \qquad (4.10)
$$

forms a nonzero absolutely continuous part for the distribution of the vector in (3.3), for any Borel set $A\in\mathcal{B}(\mathbb{R}^2)$.

(iii) Let us now assume that L^i with the triplet $(0,0,F_i(dy))$ such that $F_i(dy)$ satisfies the condition of (2.2), while $F_i(\mathbb{R})<\infty$ holds, for any $i=1,\dots,n$. In this case, L^i is a compound Poisson process which admits the representation $L_t^i=\sum_{m=1}^{N_t^i} Y_m^i$, for $t\in[0,T^*]$, where $N^i=(N_t^i)_{t\in[0,T^*]}$ is a Poisson process of intensity $\lambda_i>0$ and jump times $(\tau_m^i)_{m\in\mathbb{N}}$, and $(Y_m^i)_{m\in\mathbb{N}}$ is a sequence of mutually independent and independent of N^i identically distributed random variables with distribution $F_i(dy)/\lambda_i$, for some $i=1,\dots,n$. There is no restriction to assume that $F_i(\{0\})=0$. Let us denote by $\Psi_i(z_1,z_2;y_1,y_2)$ a version of the conditional distribution:

$$
\begin{aligned}
&\Psi_i(z_1,z_2;y_1,y_2) \qquad (4.11)\\
&= Q\left(\int_0^T f_i(t)\,dL_t^i\le z_1, \int_0^T g_i(t)\,dL_t^i\le z_2 \,\middle|\, \tau_2^i\le T<\tau_3^i, Y_1^i=y_1, Y_2^i=y_2\right)
\end{aligned}
$$

for all $(z_1,z_2)\in\mathbb{R}^2$ and some $(y_1,y_2)\in\mathbb{R}^2$ fixed. Then, by virtue of independence of N^i and (Y_1^i,Y_2^i), we get:

$$
\Psi_i(z_1,z_2;y_1,y_2) = Q\big(f_i(\tau_1^i)\,y_1+f_i(\tau_2^i)\,y_2\le z_1, g_i(\tau_2^i)\,y_1+g_i(\tau_2^i)\,y_2\le z_2 \,\big|\, \tau_2^i\le T<\tau_3^i\big) \qquad (4.12)
$$

for $(z_1, z_2) \in \mathbb{R}^2$ and $F_i(dy_1) \otimes F_i(dy_2)$-a.s.. By taking into account the fact that the vector (y_1, y_2) is unequal zero $F_i(dy_1) \otimes F_i(dy_2)$-a.s., it follows from the arguments similar to the ones used in part (ii) above that the function $\Psi_i(z_1, z_2; y_1, y_2)$ has a nonzero absolutely continuous part $F_i(dy_1) \otimes F_i(dy_2)$-a.s.. Then, because of the fact that:

$$Q\bigg(\int_0^T f_i(t)\, dL_t^i \le z_1, \int_0^T g_i(t)\, dL_t^i \le z_2 \,\bigg|\, \tau_2^i \le T < \tau_3^i \bigg) \tag{4.13}$$
$$= \int \int \Psi_i(z_1, z_2; y_1, y_2)\, F_i(dy_1) \otimes F_i(dy_2)$$

the same property holds for the conditional distribution of the vector in (3.3) under $\{\tau_2^i \le T < \tau_3^i\}$, for $i = 1, \dots, n$.

(iv) Let us finally assume that L^i has the triplet $(0, 0, F_i(dy))$ such that $F_i(dy)$ satisfies the condition of (2.2), but $F_i(\mathbb{R}) = \infty$ holds, for any $i = 1, \dots, n$. Then, for any relatively small $\varepsilon > 0$ fixed, the process L^i admits the Lévy-Itô decomposition:

$$L_t^i = J_t^{i,\varepsilon} + (L_t^i - J_t^{i,\varepsilon}) \quad \text{with} \quad L_t^{i,\varepsilon} = \sum_{0<s\le t} \Delta L_s^i \, \mathbb{1}\big(|\Delta L_s^i| > \varepsilon\big) \quad \text{and} \quad \Delta L_t^i = L_t^i - L_{t-}^i \tag{4.14}$$

for all $t \in [0, T^*]$. Here, $(J_t^{i,\varepsilon})_{t\in[0,T^*]}$ is a compound Poisson process for which the result of part (iii) above holds, while $(L_t^i - J_t^{i,\varepsilon})_{t\in[0,T^*]}$ is a limit of compound Poisson processes, for any $i = 1, \dots, n$. In this case, the desired assertion holds in its general form, because the vectors:

$$\bigg(\int_0^T f_i(t)\, dJ_t^{i,\varepsilon}, \int_0^T g_i(t)\, dJ_t^{i,\varepsilon} \bigg) \quad \text{and} \quad \bigg(\int_0^T f_i(t)\, d(L_t^i - J_t^{i,\varepsilon}), \int_0^T g_i(t)\, d(L_t^i - J_t^{i,\varepsilon}) \bigg) \tag{4.15}$$

are independent, for any $i = 1, \dots, n$. $\square$

Proof of Lemma 3.5. (i) Assume that the functions $f_i(t)$ and $g_i(t)$ are affine independent on $[0, T]$, for every $i = 1, \dots, n$. Then, by virtue of the assertion of Lemma 3.3, we get that the distribution of the vector:

$$\bigg(\int_0^T f_i(t)\, dL_t^i, \int_0^T g_i(t)\, dL_t^i \bigg) \tag{4.16}$$

has a nonzero absolutely continuous part, for every $i = 1, \dots, n$. Hence, by virtue of the independence of the processes L^i, $i = 1, \dots, n$, the distribution of the vector:

$$\bigg(\sum_{i=1}^n \int_0^T f_i(t)\, dL_t^i, \sum_{i=1}^n \int_0^T g_i(t)\, dL_t^i \bigg) \tag{4.17}$$

has an absolutely continuous part too, but the latter property cannot hold due to the condition of (3.5). Thus, we may conclude that there exist $a_i, h_i \in \mathbb{R}$ such that the representation:

$$f_i(t) = a_i\, g_i(t) + h_i \tag{4.18}$$

holds, for all $t \in [0, T]$ and every $i = 1, \dots, n$, and therefore, the expression:

$$\sum_{i=1}^n \int_0^T f_i(t)\, dL_t^i = \sum_{i=1}^n a_i \int_0^T g_i(t)\, dL_t^i + \sum_{i=1}^n h_i\, L_T^i \tag{4.19}$$

is satisfied.

(ii) We now show that one can take $h_i = 0$ in the expression of (4.18), for every $i = 1, \ldots, n$. For this purpose, let us assume that $h_k \neq 0$, for some $k = 1, \ldots, n$. Then, we observe that, by virtue of the expression in (3.5), the representation in (4.19) implies that the equality:

$$\frac{1}{h_k}\, G_n\bigg(\sum_{j=1}^{n} I_T^j \bigg) - \sum_{j=1}^{n} \frac{a_j}{h_k}\, I_T^j - \sum_{j=1, j \neq k}^{n} \frac{h_j}{h_k}\, L_T^j = L_T^k \quad (Q\text{-a.s.}) \tag{4.20}$$

holds, with some Borel function G_n, $n \in \mathbb{N}$, where we set:

$$I_T^j = \int_0^T g_j(t)\, dL_t^j \tag{4.21}$$

for every $j = 1, \ldots, n$. Thus, by virtue of the independence of the processes L^i, $i = 1, \ldots, n$, it follows from the expressions in (4.20) and (4.21) that:

$$\frac{1}{h_k}\, G_n\bigg(I_T^k + \sum_{j=1, j \neq k}^{n} u_j \bigg) - \frac{a_k}{h_k}\, I_T^k - \sum_{j=1, j \neq k}^{n} \frac{a_j}{h_k}\, u_j - \sum_{j=1, j \neq k}^{n} \frac{h_j}{h_k}\, v_j = L_T^k \quad (Q\text{-a.s.}) \tag{4.22}$$

for $Q(I_T^j, L_T^j)$-almost all (u_j, v_j), for every $j = 1, \ldots, n$, $j \neq k$. Hence, there exists at least two vectors $(u_1', \ldots, u_{k-1}', u_{k+1}', \ldots, u_n')$ and $(v_1', \ldots, v_{k-1}', v_{k+1}', \ldots, v_n')$ such that we have:

$$\frac{1}{h_k}\, G_n\bigg(I_T^k + \sum_{j=1, j \neq k}^{n} u_j' \bigg) - \frac{a_k}{h_k}\, I_T^k - \sum_{j=1, j \neq k}^{n} \frac{a_j}{h_k}\, u_j' - \sum_{j=1, j \neq k}^{n} \frac{h_j}{h_k}\, v_j' = L_T^k \quad (Q\text{-a.s.}) \tag{4.23}$$

for some $k = 1, \ldots, n$. Taking into account the fact that L^k is nondeterministic and applying the result of Lemma 3.4 for the appropriate function in the expression of (4.23), for such $(u_1', \ldots, u_{k-1}', u_{k+1}', \ldots, u_n')$ and $(v_1', \ldots, v_{k-1}', v_{k+1}', \ldots, v_n')$ fixed, we may therefore conclude that $g_k(t)$ would have to be constant on $[0, T]$, that contradicts the assumption above. The latter fact directly yields that h_k has to be zero, for every $k = 1, \ldots, n$.

(iii) We finally show that $a_1 = \cdots = a_n$ in the representation of (4.18). In this case, using the condition of (3.5) as well as the fact proved in part (ii) above that $h_i = 0$ holds, for every $i = 1, \ldots, n$, in the representation of (4.18), we get from the expression in (4.19) that the equality:

$$\widetilde{G}_{n,k}\bigg(\sum_{j=1}^{n} I_T^j \bigg) = \sum_{j=1, j \neq k}^{n} (a_j - a_k)\, I_T^j \quad (Q\text{-a.s.}) \tag{4.24}$$

holds, for any $k = 1, \ldots, n$. Here, we set $\widetilde{G}_{n,k}(x) = G_n(x) - a_k x$, for all $x \in \mathbb{R}$, and the random variables I_T^j, $j = 1, \ldots, n$, are defined in (4.21) above. Then, taking into account the independence of the processes L^i, $i = 1, \ldots, n$, it follows from the expression in (4.24) that:

$$\widetilde{G}_{n,k}\bigg(\sum_{j=1}^{n} u_j \bigg) = \sum_{j=1, j \neq k}^{n} (a_j - a_k)\, u_j \tag{4.25}$$

for $Q(I_T^j)$-almost all u_j, for every $j = 1, \ldots, n$. Thus, there exist at least $n-1$ linearly independent vectors $(u_1^m, \ldots, u_{k-1}^m, u_{k+1}^m, \ldots, u_n^m)$, $m = 1, \ldots, n-1$, such that we have:

$$\widetilde{G}_{n,k}\Bigg(u_k + \sum_{j=1, j\neq k}^{n} u_j^m\Bigg) = \sum_{j=1, j\neq k}^{n} (a_j - a_k)\, u_j^m \tag{4.26}$$

for $Q(I_T^k)$-almost all u_k, for every $m = 1, \ldots, n-1$ and any $k = 1, \ldots, n$. Hence, using the fact that L^k is nondeterministic and the function $g_k(t)$ is nonconstant on $[0, T]$ by assumption, we may conclude that there exist at least $n-1$ points u_k^m, $m = 1, \ldots, n-1$, such that the values $u_1^m + \cdots + u_n^m$, $m = 1, \ldots, n-1$, coincide with each other and the expressions in (4.26) hold with $u_k = u_k^m$, $m = 1, \ldots, n-1$, while the right-hand sides of (4.26) can represent different numbers, for any $k = 1, \ldots, n$. The latter fact implies that we must have $a_j = a_k$, for every $j = 1, \ldots, n$, $j \neq k$, and any $k = 1, \ldots, n$, in the expressions of (4.26), and therefore, concludes the proof of the lemma. $\square$

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