Sequential regular variation: extensions of Kendall's theorem
by
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David George Kendall 1918-2007, in memoriam
To John Kingman, on his 80^{th} birthday, 28 August 2019

Abstract. Regular variation is a continuous-parameter theory; we work in a general setting, containing the existing Karamata, Bojanic-Karamata/de Haan and Beurling theories as special cases. We give sequential versions of the main theorems, that is, with sequential rather than continuous limits. This extends the main result, a theorem of Kendall's (which builds on earlier work of Kingman and Croft), to the general setting.

Keywords. Kendall's Theorem, regular variation, quasi regular variation, general regular variation, uniform convergence theorem, Gołąb-Schinzel equation, Beurling-Goldie equation, essential limits, croftian theorems, category-measure duality.

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1. Introduction.

We are concerned here with *regular variation*, RV for short (for background on which we refer to the standard work [7], BGT below). This occurs first in the classical *Karamata* setting

$$\lim_{x\to\infty} f(tx)/f(x) = K(t) \qquad \text{(locally uniformly in } t \ (\forall t > 0)) \qquad (K)$$

(see e.g. BGT Ch. 1), and then in the *Bojanic-Karamata/de Haan* setting (see e.g. BGT Ch. 3),

$$[f(x+t) - f(x)]/h(x) \to K(t) \qquad \text{(locally uniformly in } t). \qquad (BKdH)$$

The basic result in the theory of regular variation is the *Uniform Convergence Theorem*, UCT (BGT, Th. 1.2.1): with f and h above Baire (i.e. having the Baire property, BP) or measurable, convergence is necessarily locally uniform, hence our general assumption both here and in the contexts below.

Next, we need the *Beurling* setting (see e.g. BGT § 2.11 and [12, 17]), using an auxiliary function φ, best implemented algebraically (see § 3 below) via the *Popa binary operation*

$$x \circ_\varphi t := x + t\varphi(x).$$

This leads to the *Beurling* theory of regular variation [17], below.

The auxiliary φ above is to satisfy

$$\eta_x(t) := \varphi(x+t\varphi(x))/\varphi(x) = \varphi(x\circ_\varphi t)/\varphi(x) \to \eta(t) \qquad \text{(locally uniformly in } t).$$

When $\varphi(x) = O(x)$, these are the *self-equivarying* functions, $\varphi \in SE$, of [51]; when specialized to the case $\varphi(x) = o(x)$ and $\eta \equiv 1$ these reduce to the classical *self-neglecting functions* [BGT, § 2.11]. Here again, it is known that, for φ Baire/measurable, the convergence is necessarily locally uniform, provided φ satisfies one of a series of four possible additional properties, including continuity or monotonicity: see [51, Th. 4]. We will need

Theorem O [51, Th. 0]. *If $\varphi(x) = O(x)$ and $\eta_x(t) \to \eta(t) = \eta^\varphi(t)$, locally uniformly in t, for $t > 0$, then η satisfies the Gołąb-Schinzel functional equation*

$$\eta(s \circ_\eta t) = \eta(s)\eta(t). \tag{GS}$$

Notational convention. In Theorem O above η_x contains the x which tends to infinity. After this passage to the limit, attention focuses on the limit function $\eta(t)$ which will depend on a parameter ρ, below. We allow ourselves to denote this limit by $\eta_\rho(t)$ and let context speak for itself here.

For φ Baire/measurable, η^φ is Baire/measurable and so is continuous (below), as are the positive solutions of (GS), the only ones of interest here, which take the form $\eta(t) = \eta_\rho(t) := 1 + \rho t$, for $t > \rho^* := -1/\rho$ with $\rho \geq 0$ (and 0 to the left of ρ^*, though here we work initially in $\mathbb{R}_+$) – see the surveys [18] and [36]; cf. [51]. So $\eta = \eta_0 \equiv 1$ yields the desired limit of $\eta_x(t)$ for the self-neglecting case.

Finally, we need the setting of *general* regular variation, recently developed in [17]:

$$[f(x \circ_\varphi t) - f(x)]/h(x) \to K(t) \qquad \text{(locally uniformly in } t), \tag{GRV}$$

with the domain of t naturally extending to $t > \rho^*$. Here f is the function of primary interest; h and φ are auxiliary functions; the limit function on the right we call the *kernel* function. The instance here $h(x) \equiv 1$ pre-dating this, developed in [12], is termed *Beurling* RV; f then is said to be φ-RV: again see § 3.

These (limit) kernel functions K satisfy *functional equations.* The classical Karamata setting yields the multiplicative *Cauchy functional equation,* (CFE) for short below. In addition to the *Gołąb-Schinzel* functional equation above, there are: the *Chudziak-Jabłońska* functional equation

$$K(u \circ_\eta v) = K(v)K(u); \tag{CJ}$$

the *Beurling-Goldie* functional equation of the general setting

$$K(u \circ_\eta v) = K(u) \circ_\sigma K(v), \tag{BG}$$

and the original *Goldie* functional equation [52] of the $\varphi \equiv 1$ setting, which it extends,

$$K(u + v) = K(u) \circ_\sigma K(v). \tag{G}$$

See § 9.6 for their continuous solutions, and a discussion of how discontinuous solutions are excluded by the 'blanket assumption of non-triviality', stated in § 3 ahead of Theorem 4 (after the necessary preliminaries in § 2).

All four of the limiting settings above involve *continuous* limits. However, *sequential* limits (see e.g. BGT § 1.9) are also important, both in theory (see the theorems below) and in applications (particularly to probability – see e.g. § 9.3 – which as it happens originally motivated the theory).

The prototypical sequential result here is due to Croft [22] in 1957. The role of the Baire Category Theorem, and the relevance to probability theory, are due to Kingman [41, 42] in 1963 and 1964, and Kendall [40] in 1968.

The Baire Category Theorem is sequential, and so its role in the sequential results here is thematic. All the 'Baire' results below need only the Axiom of Dependent Choice(s), DC. We comment briefly on the set-theoretic axiomatic background here in § 9.1.

For the interplay between category and measure in settings such as this, we refer to a number of our previous studies, for instance [12, 13, 14] and [50, 51]; it is *category rather than measure* that is primary here. For background on the axiomatics underpinning results in this area, we refer to our recent survey [16]; see again § 9.1.

We will rely on the following combinatorial tool. Below, B is 'negligible', $B \in \mathcal{N}$, will mean B is meagre or null according to context, 'quasi all' will mean 'off a negligible set', while 'non-negligible' will implicitly mean Baire/measurable (and non-meagre/non-null).

Proposition 1 (Affine Two-sets Lemma [12, Lemma 2]**).** *For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for A, B non-negligible, then for quasi all $b \in B$ there exists an infinite set $\mathbb{M} = \mathbb{M}_b \subseteq \mathbb{N}$ such that*

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A.$$

In § 2 below we review (Theorems K, K1, K2) the results we need, and prove our main result, Theorem 1, extending Kendall's Theorem to the Karamata setting, and Theorem 2, the Characterisation Theorem for 'quasi regular variation', where limits are taken avoiding a negligible exceptional set (cf. BGT § 2.9). In § 3, we turn to 'Beurling regular variation' [51], in its Baire version, proving the UCT in this setting (Th. 3) and the relevant version of Kendall's Theorem (Th. 4), involving the functional equation (CJ). We give the results we need on infinite combinatorics in § 4 (Prop. 3). In § 5 we deal with *general regular variation* (Beurling setting, Baire versions). Here, the relevant version of Kendall's Theorem (Th. 5) involves the functional equation (BG). Measure versions (Th. 1M, Th. 4M, Th. 5M) follow in § 6. Then in § 7 we turn to the regular variation of the various sequences appearing in Kendall's Theorem (Theorems 6I or 6M depending on context – hereafter Theorem 6 for brevity). Character degradation (Theorem 7) resulting from ess-lim follows in § 8 (cf. that from limsup and liminf in [9]). Complements are presented in § 9 which we close with an Appendix on the relevant aspects of coding (i.e. the links between classical and effective descriptive set theory).

2. Characterization theorems: the Karamata setting.

Below $\mathbb{R}_+ := (0, \infty)$, and functions are *Baire* if they have the Baire property, BP. We recall from [8] that a divergent sequence c_n (i.e. with $\limsup_{n\to\infty} c_n = \infty$) is said to be *additively admissible*, resp. *multiplicatively admissible,* if

$$\limsup_{n\to\infty} c_{n+1} - c_n = 0, \qquad \text{resp. } \limsup_{n\to\infty} c_{n+1}/c_n = 1.$$

In view of the results below, especially Theorem K2 and Corollary, it is appropriate to comment that Kingman's more restrictive condition:

$$\lim_{n\to\infty} c_{n+1} - c_n = 0, \qquad \text{resp. } \lim_{n\to\infty} c_{n+1}/c_n = 1, \tag{KC}$$

is necessary and sufficient for the continuous and sequential forms of regular variation to combine well (sufficiency as here and in BGT §1.9, necessity as in Weissman's lemma BGT Lemma 1.9.6). Without it, 'anything can happen'. For precise formulations of such results, see [3]. These hinge on the Doeblin universal laws [26, XVII.9].

As usual, we pass between multiplicative and additive versions at will by using the exp/log isomorphisms between the additive group $\mathbb{R}$ (Haar measure = Lebesgue measure $\mathrm{d}x$) and the multiplicative group $\mathbb{R}_+$ (Haar measure $\mathrm{d}x/x$); cf. [BGT, Ch. 1] and [17].

Theorem K (Characterization theorem of Karamata regular variation, cf. [BGT, 1.4.1]). *If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is Baire/measurable and regularly varying, that is for some function g*

$$\lim_{x\to\infty} f(tx)/f(x) = g(t) \qquad (\forall t > 0),$$

then g is Baire/measurable and multiplicative:

$$g(st) = g(s)g(t) \qquad (\forall s, t > 0), \tag{CFE}$$

and so for some $\gamma \in \mathbb{R}$

$$g(t) = t^{\gamma}.$$

Theorem K1 (Kingman's Croftian Theorem [41, 42], cf. [BGT, 1.9.1]). *Take $\{c_n\}$ additively admissible, I an open interval of $\mathbb{R}$.*
(i) *If $G \subseteq \mathbb{R}$ open and unbounded from above, then*

$$c_n + x \in G \text{ infinitely often}$$

for some $x \in I$.
(ii) *If $f : \mathbb{R} \to \mathbb{R}$ continuous and*

$$\lim_{n\to\infty} f(c_n + x) \text{ exists for all } x \in I,$$

then

$$\lim_{x\to\infty} f(x) \text{ exists.}$$

The next result is Kendall's sequential characterization theorem of regular variation.

Theorem K2 (Kendall's Theorem [40, Th. 16], cf. [BGT, 1.9.2]). *For* $\{x_n\}_{n\in\mathbb{N}}$ *multiplicatively admissible and* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *continuous: if, as* $n \to \infty$,

$$a_n f(\lambda x_n) \to g(\lambda) \qquad (\lambda \in I)$$

for some interval $I \subseteq (0, \infty)$, *positive sequence* $\{a_n\}_{n\in\mathbb{N}}$ *and continuous function* $g : I \to \mathbb{R}_+$, *then* f *is regularly varying: for each* $t > 0$,

$$K(t) := \lim_{x\to\infty} f(tx)/f(x)$$

exists, is finite, multiplicative, and both Baire and measurable. So $K(t) = t^\kappa$ *for some* κ.

The interval I here may be arbitrarily small: a smidgen's worth of sequential regular variation implies true regular variation of f, and $\{a_n\}_{n\in\mathbb{N}}$:

Corollary (cf. Theorem 6I or 6M, § 7)**.** *In Kendall's Theorem,* $\{a_n\}_{n\in\mathbb{N}}$ *is regularly varying relative to* $\{x_n\}_{n\in\mathbb{N}}$ *with index* $-\kappa$: *if* $f(x) \sim x^\kappa \ell(x)$, *with* ℓ *slowly varying, then for some constant* c

$$a_n \sim c x_n^{-\kappa}/\ell(x_n).$$

Intervals as such are not needed here: the same is true for arbitrarily small non-negligible (Baire/measurable) sets (see § 7). Notice that, given the hypotheses above, the limit function $K(t)$ is in fact the sequential limit $\lim_{n\to\infty} f(nt)/f(n)$ (of Baire/measurable functions in the former case, and continuous functions in the latter), so is Baire/measurable. The final assertions, characterizing the relevant limit function, follow from theorems concerning Baire/measurable solutions of (CFE), the Cauchy functional equation (see e.g. [10]).

Variants on the characterization theorem above are possible. First, one may drop any condition of 'topological good behaviour' (BP, or measurability, which is BP under a change from the Euclidean to the density topology; see e.g. [13]), and weaken the quantifier on t above, at the cost of imposing a side-condition (the classical prototype is the Heiberg-Seneta condition: BGT, 1.4.3) – see [15, § 7]. By contrast, here we take the passage to the limit *sequentially* as in Kendall's Theorem, through a suitable (admissible) sequence $\{x_n\}$, with our function f again appearing *once,* rather than *twice*, but allow exceptions on a meagre set. Our conclusion is of regular variation *off an exceptional set – 'quasi regular variation',* as we shall call it. This reduces to ordinary regular variation if we require also 'topological good behaviour' of the 'essential limit' (below). As in [9], [12, § 11], the passage to the essential limit results in *character degradation* (§ 7). Examination of this requires us to specify the set-theoretic axioms we use (cf. [9, 16]).

Our first definition covers both category and measure needs, again by passage to the density topology.

Definition 1. For L_f finite, say that $f(x)$ *has essential limit* L_f as $x \to \infty$ and write $f(x) \to^{\text{ess}} L_f$, or $\text{ess-lim}_{x\to\infty} f(x) = L_f$, if for each $\varepsilon > 0$ there is $X^f_\varepsilon \in \mathbb{R}$ and meagre M^f_ε such that

$$|f(x) - L_f| < \varepsilon \text{ for all } x > X^f_\varepsilon \text{ off the set } M^f_\varepsilon. \qquad (*)$$

Definition 2. Say that a Baire function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *quasi regularly varying* weakly (resp. strongly) if $g(t) := \text{ess-lim}_{x\to\infty} f(tx)/f(x)$ exists and is finite (and resp. g is Baire).

Our new results here are variants on or additions to Kendall's Theorem, particularly Theorems 1 and 6 below.

Theorem 1. *For $\{x_n\}_{n\in\mathbb{N}}$ multiplicatively admissible and $f : \mathbb{R}_+ \to \mathbb{R}_+$ Baire, if*

$$a_n f(\lambda x_n) \to g(\lambda) \qquad (\lambda \in B)$$

for some non-meagre Baire set $B \subseteq (0, \infty)$, positive sequence $\{a_n\}_{n\in\mathbb{N}}$ and function $g : B \to \mathbb{R}_+$, then f is (strongly) quasi regularly varying: for each $s > 0$,

$$K(s) := \text{ess-}\lim_{\lambda\to\infty} f(s\lambda)/f(\lambda)$$

exists and is finite, and multiplicative. As g is Baire on B, K is locally bounded near $s = 1$, and so $K(s) = s^\kappa$ for some $\kappa \in \mathbb{R}$.

Some such result was suggested by [5, footnote p. 162] in a discussion of Kendall's Theorem. The question arises of strengthening Theorem 1 by 'thinning': requiring convergence for a smaller λ-set. Such 'quantifier weakening' is possible, and involves results of Steinhaus-Weil type; see [14, 15] and § 9.5. We delay the proof of Theorem 1 to establish some preparatory results.

Lemma 1. (i) Essential limits preserve sums: if $f(x) \to^{\text{ess}} L_f$ and $g(x) \to^{\text{ess}} L_g$, then $(f + g)(x) \to^{\text{ess}} L_f + L_g$; *likewise for products.*

(ii) *If* $h(x+u)-h(x) \to^{\text{ess}} L_u$ *and* $h(x+v)-h(x) \to^{\text{ess}} L_v$, *then*

$$h(x+u+v)-h(x) \to^{\text{ess}} L_u + L_v.$$

Proof. (i) For $\varepsilon > 0$ choose $X_\varepsilon^f \in \mathbb{R}$ and meagre M_ε^f and likewise $X_\varepsilon^g \in \mathbb{R}$ and meagre M_ε^g so that $(*)$ above holds for f and g respectively. Then $(*)$ for $(f+g)$ holds (with 2ε in lieu of ε) for all $x > \max\{X_\varepsilon^f, X_\varepsilon^g\}$ off the meagre set $M_\varepsilon^f \cup M_\varepsilon^g$. Logarithmic transformation yields the analogous result for products.
(ii) With $f(x) := h(x+u)-h(x)$ and $g(x) := h(x+v)-h(x)$, since $f(x) \to^{\text{ess}} L_u$ and $g(x) \to^{\text{ess}} L_v$,

$$[h(x+u+v)-h(x+v)] + [h(x+v)-h(x)] \to^{\text{ess}} L_u + L_v,$$

that is

$$[h(x+u+v)-h(x)] \to^{\text{ess}} L_u + L_v. \qquad \square$$

Corollary. *For* $h : \mathbb{R} \to \mathbb{R}$ *Baire and* $k : \mathbb{R} \to \mathbb{R}$ *arbitrary,* $\mathbb{G}_{\text{ess}} := \{u : h(x+u)-h(x) \to^{\text{ess}} k(u)\}$ *is a subgroup, and*

$$k(u+v) = k(u) + k(v) \qquad (u, v \in \mathbb{G}_{\text{ess}}).$$

So if $\mathbb{G}_{\text{ess}}$ *contains a non-meagre Baire set, then* $\mathbb{G}_{\text{ess}} = \mathbb{R}$ *and, if* k *is Baire, then* k *is linear:* $k(u) = cu$.

Proof. That $\mathbb{G}_{\text{ess}} = \mathbb{R}$ follows here from the Subgroup Theorem for category [BGT, Cor. 1.1.4]; evidently $0 \in \mathbb{G}_{\text{ess}}$, so it suffices to note that $-u \in \mathbb{G}_{\text{ess}}$ for $u \in \mathbb{G}_{\text{ess}}$: indeed with $y = x - u$

$$h(x-u)-h(x) = -[h(y+u)-h(y)] \to^{\text{ess}} -k(u). \qquad \square$$

Theorem 2 (**Characterization of quasi regular variation).** *If* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is Baire/measurable and weakly quasi regularly varying with essential limit function* g,

$$\text{ess-}\lim_{x\to\infty} f(tx)/f(x) = g(t) \qquad (\forall t > 0),$$

then g *is multiplicative. Furthermore, if* g *is Baire (i.e.* f *is strongly quasi regularly varying), then for some* γ

$$g(t) = t^\gamma.$$

This follows from Lemma 1(ii) and its Corollary. As to the assumption that g is Baire, Theorem 7 (in § 8) clarifies the topological character of g. We turn now to a stronger form of Theorem K1(ii), which is based on our generalizations of Theorem K1(i) in [8].

Proposition 2. *For f Baire and $\{c_n\}_{n\in\mathbb{N}}$ additively admissible, if* $\lim_n f(c_n+x)$ *exists for each x in a Baire non-meagre set C, then* ess-$\lim_{x\to\infty} f(x)$ *exists, and, for quasi all $x \in C$, equals* $\lim_n f(c_n + x)$.

Proof. This follows [BGT, Th. 1.9.1(ii)]. W.l.o.g. $C := I\backslash M$ with I an open interval and M meagre. The function $\hat{f}(x) := \lim_n f(c_n + x)$ is Baire on C. By the Baire-Kuratowski Continuity Theorem (see e.g. [54, Th. 8.1]) w.l.o.g. (expanding M as necessary) $\hat{f}|(I\backslash M)$ is continuous. Fix $x_0 \in I\backslash M$; then, for any $\varepsilon > 0$, the set

$$J_\varepsilon := \{x \in C : |\hat{f}(x) - \hat{f}(x_0)| < \varepsilon\}$$

is open relative to $I\backslash M$. So w.l.o.g. $J_\varepsilon \subseteq I$ and $|\hat{f}(x) - \hat{f}(x_0)| < \varepsilon$ holds on quasi all of J_ε.

We show that $f(x) \to^{\text{ess}} \hat{f}(x_0)$. Otherwise, for some $\varepsilon > 0$ the Baire set

$$H := \{x : |f(x) - \hat{f}(x_0)| \geq \varepsilon\}$$

is essentially unbounded. Hence, by a generalization of Theorem K1 [8, Th. 3.6C], for quasi all $x \in J_\varepsilon$ there are infinitely many n with $c_n + x \in H$, i.e. $|f(c_n + x) - \hat{f}(x_0)| \geq \varepsilon$ for infinitely many n . For any such fixed $x \in J_\varepsilon$, passing to the limit yields $|\hat{f}(x) - \hat{f}(x_0)| \geq \varepsilon$, and so this holds on quasi all of J_ε. This contradicts the reverse inequality, which holds on almost all of J_ε. $\square$

Proof of Theorem 1. We work in the multiplicative positive reals, and begin by recalling a Kemperman-type Displacements Lemma ([8, Cor. p. 157] – there in additive notation) asserting that for B Baire non-meagre, $B\cap sB$ is non-meagre for all s close enough to 1 — for $s \in J_\varepsilon := ((1+\varepsilon)^{-1}, 1+\varepsilon)$, say, for some $\varepsilon > 0$, cf. Theorem 6 below. (This may also be deduced from the Pettis-Piccard Theorem, [55, 56], [BinGT, Th. 1.1.1], [10, Th. P].) Since scaling preserves category, this implies that $C(s) := B \cap s^{-1}B$ is non-meagre for any $s \in J_\varepsilon$.

Now, for the most part, we follow Kendall's proof [40, Th. 16, p. 192]: if $\lambda \in B$ and $s\lambda \in B$ (i.e. $\lambda \in C(s)$), then

$$f(s\lambda x_n)/f(\lambda x_n) = a_n f(s\lambda x_n)/a_n f(\lambda x_n) \to g(s\lambda)/g(\lambda).$$

Put $k_s(\lambda) := f(s\lambda)/f(\lambda)$; then for any fixed $s \in J_\varepsilon$, for λ in the Baire non-meagre set $C(s)$,

$$k_s(\lambda x_n) \to g(s\lambda)/g(\lambda), \text{ as } n \to \infty.$$

By Prop. 2, $K(s) := \text{ess-lim}_{x\to\infty} k_s(x)$ exists for this arbitrary $s \in J_\varepsilon$; furthermore, for each such s and quasi all $\lambda \in C(s)$,

$$K(s) = \lim_n f(s\lambda x_n)/f(\lambda x_n) = g(s\lambda)/g(\lambda).$$

As in the Corollary above, $\mathbb{G} := \{s : K(s) \text{ is well defined}\}$ is a multiplicative subgroup, since, for fixed $s, t \in \mathbb{G}$,

$$\begin{aligned} K(st) &= \text{ess-}\lim_{x\to\infty} k_{st}(x) = \text{ess-}\lim_{x\to\infty}[f(stx)/f(tx)] \cdot \text{ess-}\lim_{x\to\infty}[f(tx)/f(x)] \\ &= K(s)K(t). \end{aligned}$$

Moreover, $\mathbb{G}$ contains the interval J_ε, so, by the Steinhaus Subgroup Theorem (see e.g. [9, Th. 6.2]), $\mathbb{G} = \mathbb{R}_+$. So $\text{ess-lim}_{x\to\infty} k_s(x)$ exists for all s and is multiplicative, as in Lemma 1(ii).

For the last part, being a sequential limit g is (positive and) Baire. By passing to a smaller non-meagre subset of B, we may w.l.o.g. assume that g is bounded on B, so that, for some $0 < a < b$, say:

$$a < g(\lambda) < b \qquad (\lambda \in B).$$

Likewise, by passage to a corresponding smaller $\varepsilon > 0$, if necessary, we again conclude (as above) that, for $s \in J_\varepsilon$ and quasi all $\lambda \in C(s)$,

$$K(s) = g(s\lambda)/g(\lambda) > 0.$$

For $s \in J_\varepsilon$ choose $\lambda_s \in C(s)$ to witness the preceding equation. Then

$$K(s) = g(s\lambda_s)/g(\lambda_s) \in (a/b, b/a).$$

So K is locally bounded on J_ε. Hence, by the Darboux Theorem ([23, 24], [46, §14.4]) or the Banach-Mehdi Theorem (see e.g. [10, Th. BM]), K is continuous, and so a power function: $K(s) = s^\kappa$ for all $s > 0$, since $K > 0$. $\square$

Remarks. 1 (*'Nice' versions of Baire functions*)**.** The proof above centers on $k(s,t) := g(st)/g(t)$ as a function of two variables, with g Baire. It is instructive to take note why the composite function k may be assumed Baire.

In order for $f : \mathbb{R}^d \to \mathbb{R}$ to be a Baire function it is necessary and sufficient that the restriction $f|(\mathbb{R}^d \backslash M)$ be continuous for some meagre set M [54, Th. 8.1]. Above we opted to neglect behaviour on meagre sets, so M may as well be expanded to a union of closed nwd (nowhere dense) sets, and so for f Baire, the restriction $f|H_f$ is continuous for some dense $\mathcal{G}_\delta$ set H_f. With this in mind, note why the map $(s,t) \mapsto f(st)$ may be taken Baire: for each $q \in \mathbb{Q}$, the level set $L(q) := \{(s,t) : f(st) < q\}$ is the projection of the set

$$\{(s,t,u) : u = st\} \cap \{(s,t,u) : (s,t,u) \in H_f^3 \ \&\ f(u) < q\};$$

as the second term here is a $\mathcal{G}_\delta$, the projection $L(q)$ is analytic, and so by Nikodym's Theorem [59, Cor. 2.9.4], [39, 29.14] is again Baire. As $\mathbb{Q}$ is countable, all the rational level sets are $\mathcal{G}_\delta$ sets modulo one single union of closed nwd (nowhere dense) sets.

The moral is: we may pass to a 'version' of k which is 'nice': all its rational level sets are $\mathcal{G}_\delta$. Of course, the actual function k can be as 'nasty' as the meagre set one's set-theory admits, and that depends on one's selection of a (perhaps weak) form of the axiom of choice. See e.g. the Appendix below and [16].

2. In the last part of our proof, where we deduce the form of K, we necessarily parted company with Kendall's proof which at that point refers to Theorem K to identify K. For, Theorem 1 refers to essential, rather than ordinary, limits; nor could we apply the Corollary above to $k = \log K$, as we did not then know the topological character of K.

3. Beurling regular variation: Baire versions.

Beurling slow variation [BGT, § 2.11], relative to a function φ, was used by Beurling to prove a Tauberian theorem for Borel summability, which is not of convolution form but 'convolution-like' (see e.g. [4]; for links with Riesz means, see [6]). For *Beurling's Tauberian theorem,* extending Wiener's Tauberian Theorem, see e.g. [43, IV.11], [17, § 6.1] (there, the 'convolution-like' operation is shown to be an 'asymptotic convolution'). This has led recently to a generalization [12, 17] of RV to φ-RV, shortly to be recalled. Below, Kendall's Theorem is first extended to this context. It has emerged recently that the most convenient way to define Beurling's idea is to use some simple algebraic tools. Key here is notation introduced by Popa [57] (and later independently by Javor [37]) to study the equation (GS) above, whose central role for RV was established only quite recently. For arbitrary $h : \mathbb{R} \to \mathbb{R}$ define $\circ_h$, the (*Popa*) *circle operation* [13] on $\mathbb{R}$, as in §1 by

$$s \circ_h t = s + th(s).$$

When $h(t) = \eta_1(t) \equiv 1 + t$ this reduces to the circle operation $s \circ t = s + t + st$ of ring theory (for background see [13, § 3], [52, § 2]). In the case of $\mathbb{R}$ the operation endows it with a group structure conjugate to ordinary multiplication in view of the identity

$$s \circ t = s + t + st = (1 + s)(1 + t) - 1.$$

For η satisfying (GS), we denote the resulting *Popa (circle) group* by $\mathbb{G}_\eta := \{x : \eta(x) > 0\}$, writing its inverse operation as x_η^{-1} (the *Popa inverse*).

In this notation, as above, a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *self-equivarying*, $\varphi \in SE$, if it is $O(x)$ as $x \to \infty$ and

$$\eta_x(t) := \varphi(x \circ_\varphi t)/\varphi(x) \to \eta(t) \qquad \text{(locally uniformly in } t\text{)}.$$

By Theorem O the limit function satisfies the *Gołąb-Schinzel equation* of § 1. So $\circ_\eta$ is commutative and associative. We recall that positive solutions, relevant here, are of the form $\eta(t) = \eta_\rho(t) := 1 + \rho t$, for $t > \rho^* = -1/\rho$ with $\rho \geq 0$, and that the case $\rho = 0$ when $\eta \equiv 1$ with φ an $o(x)$ function corresponds to Beurling's original notion of a *self-neglecting* function.

Definition 3. For $\varphi \in SE$ a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *φ-regularly varying* if, for some g and all $t > 0$,

$$f(x + t\varphi(x))/f(x) \to K(t) \qquad \text{(locally uniformly in } t\text{)};$$

that is, $f(x \circ_\varphi t)/f(x) \to K(t)$.

Here again, as in the Karamata setting of regular variation of § 1 the *Uniform Convergence Theorem* holds [51, Th.1]: if $\varphi \in SE, f, K$ are all Baire/measurable, then convergence is necessarily locally uniform. Our next result extends the UCT to our present Kendall setting, and justifies below the assumption of local uniformity throughout. Here and below g, as a sequential limit, is Baire/measurable, and f is viewed as defined on $\mathbb{G}_\rho$, i.e. on (ρ^*, ∞) for the appropriate ρ.

Theorem 3 (UCT, cf. [51, Th.1] and [12, Th. 2B/M]). *Take B non-negligible, $\varphi \in SE$, f (and so $g : B \to \mathbb{R}_+$) all Baire or all measurable. If*

$$a_n f(x_n \circ_\varphi t) \to g(t) \qquad (t \in B),$$

then

$$a_n f(x_n \circ_\varphi t) \to g(t) \qquad (t \in B) \qquad \text{(locally uniformly in } t).$$

Proof. This follows from Prop. 1 (Affine Two-sets Lemma, above) as in the proof of [51, Th. 1] (cf. [12, Th. 2B/M]) with the following changes. First, replace $h_N(x \circ_\varphi t)$ by $\log f(x \circ_\varphi t)$ ('N for numerator'). Next, replace $h_D(x_n)$ ('D for denominator') by $\log(1/a_n)$. Finally, replace $\mathbb{R}$ by B. □

Blanket Assumption of non-triviality on g. We will call a function $g : B \to \mathbb{R}_+$, as above, *trivial* if it takes values only in $\{0, 1\}$.

In the corresponding functional equations in K that arise below, this in turn excludes trivial solutions (note that the multiplicative Cauchy functional equation has trivial solutions identically 0 or 1, the first excluded in Theorem 1): see § 9.6.

Theorem 4. *Take B Baire non-meagre, $\{x_n\}_{n\in\mathbb{N}}$ additively admissible, and $\varphi \in SE$ and f both Baire. If the Kendall condition*

$$a_n f(x_n \circ_\varphi t) \to g(t) \qquad (t \in B) \qquad \text{(locally uniformly in } t)$$

holds, then f is φ-regularly varying and

$$f(x + t\varphi(x))/f(x) \to K(t), \text{ for } t > 0,$$

with K continuous, satisfying the Chudziak-Jabłońska equation:

$$K(u \circ_\eta v) = K(v)K(u). \qquad (CJ)$$

Proof. Put $h_n(s) := (x_n \circ_\varphi \lambda) \circ_\varphi s$ and note that

$$\begin{aligned} h_n(s) &= (x_n + \lambda\varphi(x_n)) + s\varphi(x_n + \lambda\varphi(x_n)) \\ &= x_n + \varphi(x_n)[\lambda + s\eta_{x_n}(\lambda)] = x_n \circ_\varphi [\lambda + s\eta_{x_n}(\lambda)]. \end{aligned}$$

Also, for fixed λ and n, the function $s \mapsto h_n(s)$ is a homeomorphism of $\mathbb{R}$ under the usual (Euclidean) topology; as $\varphi(x)$ is $O(x)$,

$$(x_{n+1} \circ_\varphi \lambda) - (x_n \circ_\varphi \lambda) = (x_{n+1} - x_n) + \lambda(\varphi(x_{n+1}) - \varphi(x_n)) \to 0,$$

and so – since $\varphi(x_n + \lambda\varphi(x_n))/\varphi(x_n) \to \eta(\lambda)$ – the (bitopological) generalization of Theorem K1 [8, Th. 3.5] holds for the homeomorphisms $\{h_n\}_{n\in\mathbb{N}}$.

Now put

$$k_s(t) := f(t \circ_\varphi s)/f(t).$$

Then by the local uniformity in the Kendall condition of the theorem, because

$$\lambda + s\eta_{x_n}(\lambda) \to \lambda + s\eta(\lambda) = \lambda \circ_\eta s,$$

with η Baire (so continuous), we get

$$a_n f(x_n \circ_\varphi [\lambda + s\eta_{x_n}(\lambda)]) \to g(\lambda \circ_\eta s).$$

As before, but now working in $\mathbb{G}_\eta$, the Popa group under $\circ_\eta$, there is an interval J_ε of values s for which $C(s) := B \cap (s_\eta^{-1} \circ_\eta B)$ is non-meagre, and so for quasi all $\lambda \in C(s)$

$$\begin{aligned} k_s(x_n \circ_\varphi \lambda) &= f((x_n \circ_\varphi \lambda) \circ_\varphi s)/f(x_n \circ_\varphi \lambda) \\ &= f(x_n + \varphi(x_n)[\lambda + s\eta_x(\lambda)])/f(x_n \circ_\varphi \lambda) \\ &\to g(\lambda \circ_\eta s)/g(\lambda). \end{aligned}$$

Then, as before, by Prop. 2, $K(s) := \text{ess-lim}_{x\to\infty} k_s(x)$ exists for this arbitrary $s \in J_\varepsilon$. Also, for each such s and quasi all $\lambda \in C(s)$,

$$K(s) = \lim_n k_s(x_n \circ_\varphi \lambda) = g(\lambda \circ_\eta s)/g(\lambda) > 0,$$

and

$$K(s) = \lim_n f((x_n \circ_\varphi \lambda) \circ_\varphi s)/f(x_n \circ_\varphi \lambda) = g(\lambda \circ_\eta s)/g(\lambda).$$

Since $\eta = \eta_\rho$, for some $\rho \geq 0$, by the Steinhaus subgroup theorem, as in the Corollary, $K(s)$ exists for all s in the Popa group $\mathbb{G}_\eta := \{s : \eta(s) = 1 + \rho s > 0\}$, since K is a homomorphism. Indeed, since

$$\tilde{v}(u, x) := v[\eta(u)/\eta_x(u)] \to v,$$

and

$$x \circ_\varphi (u \circ_\eta v) = (x \circ_\eta u) \circ_\varphi \tilde{v}(u,x) = (x + \varphi(x)u) + \tilde{v}(u,x)\varphi(x + \varphi(x)u)),$$

$$\begin{aligned} K(u \circ_\eta v) &= \text{ess-}\lim_{x\to\infty} k_{u\circ_\eta v}(x) = \text{ess-}\lim_{x\to\infty} f(x \circ_\varphi (u \circ_\eta v))/f(x) \\ &= \text{ess-}\lim_{x\to\infty} [f((x \circ_\eta u) \circ_\varphi \tilde{v}(u,x))/f(x \circ_\eta u)] \cdot \text{ess-}\lim_{x\to\infty} [f(x \circ_\eta u)/f(x)] \\ &= K(v)K(u), \end{aligned}$$

giving (CJ). $\square$

4. Croftian Infinite Combinatorics

A key ingredient in the multiplicative form of Kingman's Theorem K1 is that, for $\{d_n\}_{n\in\mathbb{N}}$ multiplicatively admissible, the sequence of dilations (homeomorphisms) $h_d : x \mapsto dx$ for $d = d_1, d_2, \ldots$ has the property that, for any non-degenerate interval $J = (a,b)$, the 'tail union' $\bigcup_{n\geq m} h_{d_n}(J)$ contains an infinite half-line (cf. [8, Th. 3.2]). This property no longer holds when J is replaced by a non-null (closed) set, as in the example due to Roy Davies [8, Th. 4.6]. To circumvent this, one may replace the indexing set $\mathbb{N}$ with its natural order by the (countable) set $\mathbb{Q}_+$ of positive rationals with their natural order (induced from the reals) and employ the corresponding rational dilations $h_q(x) = qx$. Then, for any density-open set W (see below), the corresponding tail union $\bigcup_{q\geq r} h_q(W)$ contains almost all of an infinite half-line [8, Th. 3.2]. Further [8, Th. 3.2, Remark 2], this continues to hold with $\mathbb{Q}_+$ replaced by any set of dilations $\{h_d(x) : d \in D\}$ with D dense in $\mathbb{R}_+$ (equivalently: translations on $\mathbb{R}$). This may be read as saying that for fixed x, the set $\{h_d(x) : d \in D\}$ is dense in $\mathbb{R}_+$. We will also need a strengthening of this provided in the next result, which uses the $\mathbb{Q}_+$ analogue of the homeomorphisms of Th. 4 above: for fixed positive q and λ, put

$$h_q(s) := (q \circ_\varphi \lambda) \circ_\varphi s = q + \lambda\varphi(q) + s\varphi(q + \lambda\varphi(q)),$$

to be called the 'φ-*dilations*'; below we assume φ is continuous and again rely on Prop. 1 (but cf. § 9.7).

We recall that a set is *density-open* if all its points are (Lebesgue) density points. To maintain category-measure duality, for convenience we are content to adopt the following

Definition 4. Call a set *category-open* if it takes the form of an open set less a meagre set.

Whilst the correct duality stance would be to work bitopologically and use the category and measure versions of Hashimoto topologies [14], all we need below is that the intersection of two sets of one of these two types is again of the same type.

Proposition 3. *For A, B category/density open with B unbounded and φ continuous with $\varphi = O(x)$, there are arbitrarily large rationals q and points $a_q \in A$, $b_q \in B$ with*

$$h_q(a_q) = b_q.$$

Hence the tail union $\bigcup_{q \geq r} h_q(A)$ contains quasi all of an infinite half-line.

Proof. We consider only the density-open case, as the category-open case is similar but simpler. Our aim is to establish the relation

$$q + \lambda\varphi(q) + a_q\varphi(q + \lambda\varphi(q)) = b_q,$$

as above, or equivalently

$$1 + a_q m_\lambda(q) = \frac{b_q}{q + \lambda\varphi(q)} \text{ for } m_\lambda(x) := \frac{\varphi(x + \lambda\varphi(x))}{x + \lambda\varphi(x)} > 0.$$

As $\varphi(x) = O(x)$, $m_\lambda(x)$ remains bounded as $x \to \infty$, say by $M(\lambda)$. Fix $a \in A$ arbitrarily, then choose $b \in B$ as large as desired with $b > 1 + aM(\lambda)$. Since $\varphi(x)$ and $m_\lambda(x)$ are continuous in x, and $x + \lambda\varphi(x) = x(1 + \lambda\varphi(x)/x) \to \infty$, there exists x (which is as large as desired for b large enough) with

$$\frac{b}{x + \lambda\varphi(x)} = 1 + am_x(\lambda) : \qquad a = \frac{b - (x + \lambda\varphi(x))}{m_\lambda(x)(x + \lambda\varphi(x))}.$$

Fix such an x, and choose a rational sequence $q_n \to x$. Again, by the continuity of both φ and m_λ,

$$c_n := \frac{1}{m_\lambda(q_n)[q_n + \lambda\varphi(q_n)]} \to c := \frac{1}{m_\lambda(x)[x + \lambda\varphi(x)]} > 0.$$

So

$$a/c = \left(b - \frac{1}{cm_\lambda(x)}\right) : \qquad a/c \in B_0 := (A/c) \cap \left(B - \frac{1}{cm_\lambda(x)}\right).$$

Here B_0 is dense-open, since a/c is a density point both of A/c and of the translate of B; furthermore, $cB_0 \subseteq A$. Put

$$z_n := c_n \left(\frac{1}{cm_\lambda(x)} - [q_n + \lambda\varphi(q_n)] \right).$$

Then, since $cm_x(\lambda) = 1/(x + \lambda\varphi(x))$ and $m_\lambda(q_n)[q_n + \lambda\varphi(q_n)] = \varphi(q_n + \lambda\varphi(q_n))$,

$$\begin{aligned} z_n &= \frac{1/m_\lambda(q_n)}{q_n + \lambda\varphi(q_n)} \left([x + \lambda\varphi(x)] - [q_n + \lambda\varphi(q_n)] \right) \\ &= \frac{1}{\varphi(q_n + \lambda\varphi(q_n))} \left([x - q_n] + \lambda[\varphi(x) - \varphi(q_n)] \right) \to 0. \end{aligned}$$

By the Affine Two Sets Lemma above (applied to A and B_0, rather than A and B), for almost all $b_0' \in B_0$ the sequence

$$c_n b_0' + z_n \in A \text{ i.o.}$$

In particular, as $B_0 \subseteq B - 1/cm_\lambda(x)$, there are $a' \in A$, $b_0' \in B_0$, $b' \in B$ with $b_0' = [b' - 1/cm_\lambda(x)]$, and some n with

$$c_n(b' - \frac{1}{cm_\lambda(x)}) + z_n = a'.$$

Substituting for z_n gives

$$a' = c_n \left(b' - \frac{1}{cm_\lambda(x)} \right) + c_n \left(\frac{1}{cm_\lambda(x)} - [q_n + \lambda\varphi(q_n)] \right),$$

that is

$$a' = \frac{1/m_\lambda(q_n)}{q_n + \lambda\varphi(q_n)} \left(b' - [q_n + \lambda\varphi(q_n)] \right) : \qquad 1 + a' m_\lambda(q_n) = \frac{b'}{q_n + \lambda\varphi(q_n)}.$$

The final assertion follows verbatim as in [8, Th. 3.2]. $\square$

5. General regular variation: Beurling-Baire versions.

An analysis similar to that in Theorem 4 may be performed for the general setting of regular variation of § 1, i.e. with asymptotics defined by

$$[f(x + t\varphi(x)) - f(x)]/h(x) \to g(t) \qquad \text{(locally uniformly in } t\text{).}$$

The third function here, h (assumed positive), must satisfy (see the Remark below)

$$h(x \circ_\varphi t)/h(x) \to r(t) \qquad \text{(locally uniformly in } t\text{)},$$

so that, by Theorem 4, $r(t)$ satisfies (CJ).

Theorem 5. *Take $B \subseteq (0,\infty)$ Baire non-meagre, $\{x_n\}_{n\in\mathbb{N}}$ additively admissible, and $\varphi \in SE$, f (and so $g : B \to \mathbb{R}_+$) all Baire. If the Kendall condition*

$$a_n[f(x_n + t\varphi(x_n)) - f(x_n))/h(x_n) \to g(t) \qquad \text{(locally uniformly in } t\text{)}.$$

holds, then

$$[f(x + t\varphi(x)) - f(x)]/h(x) \to K(t) \qquad \text{(locally uniformly in } t\text{)},$$

with K continuous, satisfying the Beurling-Goldie equation

$$K(u \circ_\eta v) = K(u) \circ_\sigma K(v), \tag{BG}$$

and with the σ in the $\circ_\sigma$ above satisfying

$$\sigma(K(u)) = r(u): \qquad \sigma(t) = r(K^{-1}(t)).$$

Proof. Here one takes

$$k_s(t) := [f(t \circ_\varphi s) - f(t)]/h(t).$$

The analysis is unchanged (with $\tilde{v}(u,x) \to v$ in the same notation), but the kernel function K now satisfies

$$\begin{aligned}
K(u \circ_\eta v) &= \text{ess-}\lim_{x\to\infty} k_{u\circ_\eta v}(x) = \text{ess-}\lim_{x\to\infty} [f(x \circ_\varphi (u \circ_\eta v)) - f(x)]/h(x) \\
&= \text{ess-}\lim_{x\to\infty} \{[f((x \circ_\eta u) \circ_\eta \tilde{v}(u,x)) - f(x \circ_\eta u)]/h(x \circ_\eta u)\} \cdot \text{ess-}\lim_{x\to\infty} \frac{h(x \circ_\eta u)}{h(x)} \\
&\quad + \text{ess-}\lim_{x\to\infty} [f(x \circ_\eta u) - f(x)]/h(x) \\
&= K(v)r(u) + K(u).
\end{aligned}$$

Here K is a homomorphism between the two Popa groups $\mathbb{G}_\eta$ and $\mathbb{G}_\sigma$. $\square$

Remark. Notice that the preceding equation holds *if and only if* h has the asymptotic behaviour specified above. We also note that a non-zero K will necessarily be monotone – see [52].

6. Measure versions.

The aim is to read off measure analogues of Theorems K2, 1, 4 and 5 by replacing the Euclidean topology by the density topology. Mutatis mutandis, the 'density open' version of Prop. 3 allows precisely this.

Proposition 2M (cf. [8, Th. 4.1]). *For f measurable, if* $\lim_{q\in\mathbb{Q},q\to\infty} f(qx)$ *exists for each x in a non-null measurable set B, then* $\text{ess-lim}_{x\to\infty} f(x)$ *exists, and, for almost all $x \in B$, equals* $\lim_{q\in\mathbb{Q},q\to\infty} f(qx)$.

Proof. The same proof as in Prop. 2 works with $\hat{f}(x) := \lim_{q\in\mathbb{Q},q\to\infty} f(qx)$, which is measurable on B. By the Luzin Continuity Theorem (see e.g. [54, Th. 8.2]) we may assume that $\hat{f}|B$ is continuous (otherwise pass to a non-null subset B' of B on which this holds, removing a part of measure as small as desired). From here the proof is the same save that 'for infinitely many $n \in \mathbb{N}$' is replaced by 'on some unbounded sequence of $q \in \mathbb{Q}_+$'. □

This allows for the following measure versions. The first needs only ordinary dilations $\{h_q : q \in \mathbb{Q}_+\}$; the second needs the φ-dilations of § 4.

Theorem 1M. *For $f : \mathbb{R}_+ \to \mathbb{R}_+$ measurable, if*

$$a_q f(q\lambda) \to g(\lambda) \qquad (\lambda \in B) \qquad (q \to \infty \text{ through } \mathbb{Q}_+)$$

for some non-null measurable set $B \subseteq (0,\infty)$ and function $g : B \to \mathbb{R}_+$, then f is weakly almost regularly varying: for each $s > 0$,

$$K(s) := \text{ess-}\lim_{\lambda\to\infty} f(s\lambda)/f(\lambda)$$

exists and is finite, and multiplicative. As g is measurable on B, K is locally bounded near $s = 1$, and so $K(s) = s^\kappa$ for some κ.

Theorem 4M. *Take $\varphi \in SE$ continuous, $B \subseteq (0,\infty)$ measurable non-null and f (and so $g : B \to \mathbb{R}_+$) measurable. If the Kendall condition*

$$a_q f(q \circ_\varphi t) \to g(t) \qquad (t \in B,\ q \to \infty,\ q \in \mathbb{Q}) \qquad (\text{locally uniformly in } t)$$

holds, then f is φ-regularly varying and

$$f(x + t\varphi(x))/f(x) \to K(t), \text{ for } t > 0,$$

with K continuous, satisfying the equation:

$$K(u \circ_\eta v) = K(v)K(u). \tag{CJ}$$

Proof. Put $h_q(s) := (q \circ_\varphi \lambda) \circ_\varphi s$ and apply the density-open variant of Prop. 3. So the (bitopological) generalization of Theorem K1 [8, Th. 3.5M] holds for the homeomorphisms $\{h_q\}_{q \in \mathbb{Q}_+}$. $\square$

The measure analogue Theorem 5M of Theorem 5 follows similarly.

7. Regular variation: sequences.

Theorem 6I below ('I for interval') brings out a property of regular variation in the sequence $\{a_n\}_{n \in \mathbb{N}}$ of Kendall's Theorem. This (which is actually left implicit in [40]) is important in various contexts, particularly in probability theory; see e.g. [5] and § 9.3 and 9.4 below. The reduction below to a power function brings the Kendall condition into alignment with one due to Seneta, cf. BGT § 1.9.3. Below, without loss of generality we assume that g is continuous on B (by the Baire-Kuratowski or the Luzin Continuity theorems – passing to a smaller set, as necessary).

We first need to isolate from the proof of Kendall's Theorem all the available information about the g function occurring in that theorem. As usual 'negligible' below refers to meagre/null sets.

Definition. For B non-negligible, say that the continuous function $g : B \to \mathbb{R}_+$ satisfies the *Restricted Cauchy functional equation,* $(ResCFE)$ on B, if

$$g(s\lambda) = s^\kappa g(\lambda) \qquad (\forall \lambda \in (B \cap Bs^{-1}) \backslash M(s) \textit{ with } M(s) \in \mathcal{N}, \forall s \textit{ with } B \cap Bs^{-1} \notin \mathcal{N}). \tag{$ResCFE_B$}$$

This is novel here; for (conditional) functional equations, i.e. with restricted domains, see [1, Ch. 6, 7, 16], [45], [46, §13.6], and for further background literature [58]. Our result below applies to both senses of negligibility.

Theorem 6I. (i) *For I an interval, M negligible and $B = I \backslash M$ with g satisfying $(ResCFE)$ on B, there is $\lambda_0 \in B$, such that with $c := g(\lambda_0)/\lambda_0^\kappa$,*

$$g(\lambda) = c\lambda^\kappa \qquad (\lambda \in B);$$

(ii) *With* g *and* B *as above,* f, $\{x_n\}_{n\in\mathbb{N}}, \{a_n\}_{n\in\mathbb{N}}$ *as in Kendall's Theorem, i.e.*

$$a_n f(\lambda x_n) \to g(\lambda) \qquad (\lambda \in B),$$

f *is regularly varying with index* κ:

$$f(x) \sim x^\kappa \ell(x)$$

with ℓ *slowly varying.*
(iii) *In* (ii), *the sequence* $\{a_n\}_{n\in\mathbb{N}}$ *is regularly varying relative to* $\{x_n\}_{n\in\mathbb{N}}$ *with index* $-\kappa$:

$$a_n \sim cx_n^{-\kappa}/\ell(x_n).$$

Proof. (i) We denote by $M(s)$ the exceptional negligible λ-set appearing in $(ResCFE)_B$. W.l.o.g. these are meagre $\mathcal{F}_\sigma$ in the category case.

We begin in (a) below by proving a local version of (i).

(a) Here we take $I := (a,b)$ with $0 < a < b$ (and $B = I\backslash M$, with M negligible). By assumption g is continuous on B. Fix ε with

$$0 < \varepsilon < \left(\sqrt{b/a}\right) - 1.$$

Then $a(1+\varepsilon) < b/(1+\varepsilon)$, and so the interval

$$I_\varepsilon := (a(1+\varepsilon), b/(1+\varepsilon)) = \bigcap\nolimits_{s\in J_\varepsilon}(I \cap s^{-1}I) \qquad \text{for } J_\varepsilon := (1/(1+\varepsilon),(1+\varepsilon))$$

is non-degenerate. So, for $s \in J_\varepsilon$: $I_\varepsilon \subseteq I \cap s^{-1}I$, in particular, if $\lambda_0 \in I_\varepsilon$, then $\lambda_0 \in s^{-1}I$. But (adapting the notation used earlier for $C(s)$)

$$\begin{aligned} B \cap s^{-1}B &= (I\backslash M) \cap s^{-1}(I\backslash M) = (I\backslash M) \cap (s^{-1}I\backslash s^{-1}M) \\ &\supseteq C(s) := I_\varepsilon\backslash(M \cup s^{-1}M). \end{aligned}$$

By $(ResCFE)_B$, for $\varepsilon > 0$ small enough and all $s \in J_\varepsilon$, for quasi all $\lambda \in C(s)$

$$s^\kappa = g(s\lambda)/g(\lambda) \ = K(s), \tag{†}$$

the notation K following Theorem 1. Let $D = \{d_n\}_{n\in\mathbb{N}}$ enumerate a countable set dense in J_ε. The sets $d_m^{-1}M, d_m^{-1}M(d_m)$ being negligible (meagre $\mathcal{F}_\sigma$, in the category case), the set

$$H := \bigcap\nolimits_m I_\varepsilon\backslash[d_m^{-1}M \cup d_m^{-1}M(d_m)] \subseteq I$$

is non-negligible and so non-empty (in the category case: a dense $\mathcal{G}_\delta$ in I_ε, so the Baire Category Theorem applies). Take $\lambda_0 \in H$.

As above, $\lambda_0 \in d_m^{-1}I$, since $\lambda_0 \in I_\varepsilon$, i.e. $\lambda_0 d_m \in I$. Also $\lambda_0 d_m \notin M$, i.e. $\lambda_0 d_m \in I\backslash M$. So g is continuous at each $\lambda_0 d_m$. Furthermore, $\lambda_0 d_m \in I\backslash M(d_m)$, so by (†)

$$d_m^\kappa = g(\lambda_0 d_m)/g(\lambda_0) : \qquad g(\lambda_0 d_m) = d_m^\kappa g(\lambda_0).$$

But $\{d_m\}_m$ is dense in the interval J_ε, and also, as we have seen, $\lambda_0 d_m \in I\backslash M$. So, by continuity of g on B, passage to the limit gives, for all $\lambda_0 t$ in $\lambda_0 J_\varepsilon \cap B$, i.e. for quasi all t in $\lambda_0^{-1}J_\varepsilon$, that

$$g(\lambda_0 t) = t^\kappa g(\lambda_0).$$

Writing $\lambda = \lambda_0 t$, for quasi all $\lambda \in \lambda_0 J_\varepsilon \cap B$,

$$g(\lambda) = \lambda^\kappa g(\lambda_0)\lambda_0^{-\kappa} = c\lambda^\kappa.$$

(b) The argument in (a) above may be repeated, mutatis mutandis, in any subinterval of I, and this now allows us to prove (i). We put $h(x) := g(x)x^{-\kappa}$ and

$$J := \{x \in B : (\exists k_x)(\exists \delta > 0)\ h|(xJ_\delta) = k_x\}.$$

Then J is open in B, and by the earlier argument everywhere dense in I. Consider any maximal interval $J' := (a', b')$ with $J' \cap B$ contained in J and let $k = h|(J' \cap B)$. Suppose that b' is interior to I. For s with $s < 1$ the interval $s^{-1}J'$ contains $b'(1, s^{-1})$ and so meets all the maximal intervals (c, d) of J sufficiently close on the right of b'. Fix any such interval (c, d). So $s^{-1}B \cap B \supseteq s^{-1}J \cap J \supseteq s^{-1}(J' \cap B) \cap (c, d)$. Select $s\lambda \in J' \cap B$ with $\lambda = s^{-1}(s\lambda) \in (c, d) \cap B\backslash M(s)$. Let k' be the constant value of $h|(c, d) \cap B$. Then, as $\lambda \in (B \cap s^{-1}B)\backslash M(s)$,

$$g(s\lambda) = s^\kappa g(\lambda) : \qquad k = h(s\lambda) = h(\lambda) = k'.$$

Thus, on any maximal interval sufficiently close to b', $h = k$; this contradicts the maximality of (a', b') unless b' is not an interior point of I. So $b' = b$, the right end-point of I. Likewise, $a' = a$, the left end-point of I. So $h(x)$ is constant on B. $\square$(i).

(ii) In the proof of Theorem 1 in §2 we showed that for some κ the function g satisfies ($ResCFE$) on B and that f is regularly varying with index that κ. $\square$(ii).

(iii) By (ii) write $f(x) \sim x^\kappa \ell(x)$, for some $\ell \in SV$, and by (i) $g(\lambda) = c\lambda^k$ for $\lambda \in B$. So, for any fixed $\lambda \in B$,

$$a_n \sim g(\lambda)/f(\lambda x_n) \sim c\lambda^\kappa/[\lambda^\kappa x_n^\kappa \ell(\lambda x_n)] = cx_n^{-\kappa}/\ell(x_n). \qquad \square$$

Corollary. *For B Baire and g satisfying $(ResCFE)$ on B, there is a discrete family of intervals I each with a corresponding point $\lambda_I \in I$, such that with $c_I := g(\lambda_I)/\lambda_I^\kappa$,*

$$g(\lambda) = c_I \lambda^\kappa \qquad (\lambda \in I).$$

Proof. By Theorem 6I, the quasi-interior B^q of B may be represented as a union of open intervals on each of which $h(x) := g(x)x^{-\kappa}$ is constant. Consider the family of maximal open intervals in B^q on which h is constant. Then, as in the proof of (i), both end points of such a maximal interval are not limits of other maximal intervals. $\square$

So far we have considered the general Baire and the interval-minus-null cases. We turn now to the general measure variant: this addresses the measure case, and identifies a scenario of h-constancy on certain 'rational skeletons', dependent on the points of B. See [8, Th. 4.2] for the constancy of rationally invariant functions (i.e. when $h(qx) = h(x)$ for all $q \in \mathbb{Q}$). Here differences arise between Theorem 6M below and 6I because Theorem 6M requires *quantitative* measure theory (rather than *qualitative* measure theory, which is closely aligned with the Baire case). The breakdown of the usual category-duality occurs here since a measurable set need not be 'locally co-null' at any point (i.e. never meets almost all of some interval); the analogous qualitative argument delivers less information.

In the Corollary above, one has multiple constancy. Theorem 6M below in the case when B is non-null and nowhere dense opens a similar possibility.

Theorem 6M. *Take B non-null closed and g satisfying $(ResCFE)$ on B.*
(a) *If B is nowhere dense, then for almost all $b \in B$ there exists in $\mathbb{Q}$ a sequence $q_n = q_n(b) \to 1$ with $q_n b \in B$ and*

$$g(q_n b) = q_n^\kappa g(b) : \qquad g(bq_n(b))/(bq_n(b))^\kappa = g(b)/b^\kappa,$$

i.e. $h(x) := g(x)/x^\kappa$ remains constant on a rational sequence of dilations $q_n(b)$. The sequence can be selected so that $q_{2n-1}(b) \downarrow 1$ and $q_{2n}(b) = 1/q_{2n-1}(b) \uparrow 1$.

(b) *If B is somewhere dense, then $B = B_1 \cup B_0$ with B_1 open and B_0 nowhere dense; then Theorem 6I(i)-(iii) applies mutatis mutandis to B_1, and (a) above applies to B_0 if B_0 is non-null.*

Proof. We follow the notation of Th. 6I, in particular, we write $J_\Delta := [(1+\Delta)^{-1}, (1+\Delta)]$.

(a) W.l.o.g. B is (closed and) of finite measure and g is continuous on B. Now notice that the set

$$S(B) := \{\lambda \in B : (\exists \{s_n\} \uparrow 1)[g(s_n\lambda) = s_n^\kappa g(\lambda) \& (\lambda \in s_n^{-1}B)]\}$$

is analytic, hence measurable. We will show that $S(B)$ is non-empty, and hence is almost all of B: indeed, suppose otherwise, then $B \backslash S(B)$ is non-null. Then passing to a non-null closed subset, F say, it follows that $\emptyset \neq S(F) \subseteq F \cap S(B)$, contradicting that F is disjoint from $S(B)$.

Notice that density-open subsets meet in a density-open subset (empty or otherwise).

Fix p with $1/2 < p < 1$ (e.g. $p = 3/4$). We now define an operator Δ on non-empty density-open sets $A \subseteq B$. Choose a density point $b \in A$. There is $\Delta = \Delta(A) > 0$ such that

$$|A \cap bJ_\Delta| > p|bJ_\Delta|.$$

So taking $q = 1 - p$

$$|bJ_\Delta \backslash A| < q|bJ_\Delta|.$$

For $s > 1$ with $s \in J_\Delta$ one has $s - 1 < \Delta$, and $0 < 1 - s^{-1} < 1 - 1/(1+\Delta) = \Delta/(1+\Delta)$. Take $L(\Delta) := 1+\Delta = \max\{(1+\Delta), \Delta/(1+\Delta)\}$, so that $L(\Delta) \to 1$ as $\Delta \to 0$. Now for $s \in J_\Delta$

$$|bJ_\Delta \backslash bJ_\Delta s^{-1}| \leq |1 - s|bL(\Delta).$$

So for $s \in J_\Delta$,

$$|As^{-1} \cap A \cap (bJ_\Delta \backslash A)| > (p - q)|bJ_\Delta| - |1 - s|bL(\Delta).$$

So in the nhd bJ_Δ of b, for any s close enough to 1, and on either side of 1, the set $C(A) := (As^{-1} \cap A \cap bJ_\Delta)$ is density-open, qua intersection of density-open sets. As it has non-null measure it is non-empty, so contains a density point, $c(A)$ say.

We work inductively. Base step: taking C_0 to be the density-open interior of B (with $b \in C_0$), put $\Delta_1 := \Delta(C_0)$, with Δ the operator above. Take $s_1 \in J_{\Delta(1)}$ small enough, as above, so that $C_1 := C(C_0)$ is density-open and contains b.

Continue selecting $s_1, s_2, ... \to 1$, with s_{2i} increasing and s_{2i+1} decreasing and rational, and non-empty density open sets $C_i \subseteq B$ with distinguished member $b_i = c(C_i)$, so that with $\Delta(i) := \Delta(C_i)$:

i) $C_{i+1} := C(C_i) = (C_i s_i^{-1} \cap C_i \cap b_i J_{\Delta(i)})$ is non-empty and density-open so contains a density point b_{i+1};

ii) $\lambda(s_i) \in C_i \backslash \bigcup_{j \leq i} M(s_j)$.

Then $\lambda(s_i) \in (C_i s_i^{-1} \cap C_i \cap b_i J_{\Delta(i)}) \subseteq (C_j s_j^{-1} \cap C_j \cap b J_{\Delta(i)}) \subseteq B$ and so $\lambda(s_i) s_j \in C_j \subseteq B$. Thus $\lambda(s_i) \in B \cap B s_j^{-1}$. To apply $(ResCFE)_B$, note that also $\lambda(s_i) \notin M(s_j)$; so

$$g(s_j \lambda(s_i)) = s_j^{\kappa} g(\lambda(s_i)).$$

But $\lambda(s_i) \in B$, so $\lambda_0 = \lim_n \lambda(s_n)$, and $\lambda_0 s_j = \lim_n \lambda(s_n) s_j \in B$, as B is closed. By continuity of g

$$g(s_j \lambda_0) = s_j^{\kappa} g(\lambda_0).$$

(b) If B is (closed and) somewhere dense, take B_1 the union of maximal open intervals contained in B. Then $B_0 := B \backslash B_1$ is closed and nowhere dense (otherwise, it would contain an open interval disjoint from B_1). The remaining assertions are clear. □

The results above extend to general regular variation. The proof is much as above, via the Popa circle groups, with $s^{\kappa} = g(\lambda s)/g(\lambda)$ and $(B \cap s^{-1} B)$ above replaced by $K(s) = g(\lambda \circ_\eta s)/g(\lambda)$ and $(B \cap s_\eta^{-1} \circ_\eta B)$, respectively; we omit the details.

Remark. The key idea of Theorem 6 is the embedding of a specific countable set, one that is dense in itself, into an open set 'punctured' by the removal of a small or negligible part. Embeddings of countable sets by translation go back to Marczewski [47]; see [49] for recent developments.

8. Character degradation from ess-lim.

8.1 In what follows, we will need to distinguish between (general) sets of reals, and sets which can be defined by suitable coding. For background here, see e.g. the monograph Kechris [39, Ch.V] on the analytical hierarchy (note [39, V.40B] on classical v. effective descriptive set theory), and our recent survey [16]. For a deeper analysis of coding see [63, II.1.1, 25-33]; a minimal amount is in [27, § 2, p. 93]. We defer further discussion of these matters (including the ambiguous analytical class $\mathbf{\Delta}^1_2$) to the Appendix below, and to the proof of Theorem 7 below.

To say that $L = \text{ess-lim}_{x\to\infty} f(x)$ requires the assertion that *there exists* a (meagre, exceptional) *set* off which *for all x real* (large enough) $f(x)$ is as close to L as desired. In brief this has an $\exists\forall$ quantifier block in regard to the 'analytical objects': *sets* and *reals*.

It is also true that its negation $L \neq \text{ess-lim}_{x\to\infty} f(x)$, the assertion that *there exists* a (non-meagre) *set* on which *for all x real* (large enough) $f(x)$ avoids being sufficiently close to L, also has an $\exists\forall$ structure in regard to analytical objects.

These two observations have a rigorous formulation below, which adds to earlier considerations of the character of limits in Karamata and Beurling RV noted already in [9, 12]. Though our proof is largely self-contained, we refer for background and for the notation of the analytical hierarchy needed here to [39, Ch. V], [16], and to the Appendix below.

Theorem 7 (Character degradation). *For k Borel, the predicate*

$$K(s) := \text{ess-}\lim_{x\to\infty} k(s,x)$$

is of ambiguous analytical class $\mathbf{\Delta}^1_2$.

Proof. For simplicity, we consider the equation $L = \text{ess-lim}_{x\to\infty} f(x)$ with f Baire, say. This is equivalent to the predicate

$$(\exists a \in \mathbb{R})(\forall x \in \mathbb{R})(\forall m, n \in \mathbb{N})(\exists p \in \mathbb{N})\Phi(f, a, x, m, n, p),$$

where the matrix Φ is

$$[G(a(n)) \text{ is everywhere dense}]\&[x \in G(a(n))\&x > p \Rightarrow |f(x) - L| < 1/m].$$

Here Φ 'says' that, on the dense $\mathcal{G}_\delta$ set $\bigcap_n G(a(n))$ and to the right of p, the values $f(x)$ are to within $1/m$ of L (with regard to the open sets $G(.)$ see the Appendix); here $a(n) := a \cap \{1 \cdot 2^n, 3 \cdot 2^n, 5 \cdot 2^n, ...\}$, or equivalently the binary indicator sequence (so a real number) coding that set. Apart from the *arithmetic* (= natural number) quantifiers acting on Φ there are two quantifiers, the first existential, the second universal, ranging over the analytical objects of type 1, the real numbers a and x; in view of the opening *analytical* quantifier block $\exists\forall$ of 2 quantifiers over type 1 objects, the statement is said to be $\Sigma^1_2(f)$ – the parentheses acknowledge use of f as an input. The (light-faced, here) sigma symbol identifies the first quantifier as existential. Under our simplifying assumption of § 2 (Remark 2) that a Baire f is replaced by a 'nice' Borel version with all its rational level sets being $\mathcal{G}_\delta$, we can think of Φ as written not with the use of f but instead in terms of two codes (= real numbers) b and c, with each $b(m)$ and $c(m)$ as above. For more details see the Appendix. To indicate an implied need for some real parameters/codes, we use a bold symbol, and say more simply that the statement is $\boldsymbol{\Sigma}^1_2$.

As noted earlier, the negation can also be expressed as a $\Sigma^1_2(f)$ statement: the inequality $L \neq$ess-$\lim_{x\to\infty} f(x)$ is equivalent to a statement of the form

$$(\exists p \in \mathbb{N})(\exists a, b \in \mathbb{R})(\forall x \in \mathbb{R})(\forall n \in \mathbb{N})\Psi(f, a, b, x, n, p),$$

where the matrix Ψ is the conjunction of the two statements

$$[G(b) \text{ unbounded } \&[G(a(n)) \text{ everywhere dense on } G(b)]]$$

and

$$[[x \in G(a(n))\&x \in G(b)] \Rightarrow |f(x) - L| \geq 1/p],$$

saying that $|f(x) - L| \geq 1/p$ for x in the $\mathcal{G}_\delta$ set $\bigcap_m G(a(m))$ which is unbounded and non-meagre on $G(b)$.

We summarize the above by saying that ess-lim is of *ambiguous analytical class* $\boldsymbol{\Delta}^1_2$.

The analysis above extends with only a little extra complication to cover the bivariate case $L(s) =$ess-$\lim_{x\to\infty} k(s, x)$. $\square$

Remark. Character degradation under ess-lim here amounts (for f Borel) to the Borel set

$$H_m(f) := \{x : |k(x) - L| < 1/m\}$$

rising to the second level of the analytical hierarchy and becoming a more complex set: a $\boldsymbol{\Delta}^1_2$ set. (With regard to the set $H_m(f)$ see the Appendix.)

8.2. *Provably* $\boldsymbol{\Delta}^1_2$ *sets.* A subset $A \subseteq \mathbb{R}$ is said to be *provably* Δ^1_2 if there are two Σ^1_2 predicates $\Phi(x, y)$ and $\Psi(x, y)$ and a real number b such that, for all $a \in \mathbb{R}$, $a \in A$ iff $\Phi(a,b)$ and likewise $a \notin A$ iff $\Psi(a,b)$, with both these equivalences provable in the axiom system $ZF + DC$, where DC stands for the Axiom of Dependent Choices. Thus A is in $\boldsymbol{\Delta}_2^1$ (bold-faced, because of the parameter b).

Fenstad and Normann [27] noticed that a key step in [63] (in which Solovay constructs a model of set theory wherein DC holds and all subsets of $\mathbb{R}$ are Lebesgue measurable) may be re-read to show that all *provably* Δ_2^1 subsets are measurable: see Remarks 1 and 3 in [27, p. 95]. Ultimately, the argument relies on the notion of forcing provided by the partially ordered set comprising all of the Borel sets lying in a fixed countable model of set theory – see [16, § 6.1]. By using the category variant of this partial order, much the same argument gives that all *provably* $\boldsymbol{\Delta}_2^1$ subsets have the Baire property: see [38, § 14.4, p.180].

In conclusion: we should not be surprised that 'nice versions' of Baire functions yield corresponding essential-limit functions that are Baire.

9. Complements.

9.1 *Axiomatics: set-theoretic foundations for RV.* We have stressed in the Introduction the role of the Axiom of Dependent Choice(s), DC. Its great strength, as Solovay [63, p. 25] points out, is that it is sufficient for the establishment of Lebesgue measure, i.e. including its translation invariance and countable additivity ("...positive results ... of measure theory..."), and may be assumed consistently with such additional axioms as LM (*all* subsets (of $\mathbb{R}$) are Lebesgue measurable) and PB (*all* subsets have the Baire property, BP). To generate non-measurable sets one needs the Axiom of Choice AC. While the Zermelo-Fraenkel(-Skolem) axiom system ZF is common ground in mathematics, AC is not, and alternatives to it are widely used, including the two we have just mentioned. For a thorough discussion of alternatives we refer to [16], especially §10 therein.

In the standard Karamata setting of RV, continuous limits of functions may be replaced by sequential limits (as in § 2 above), so that starting with continuous functions one remains within the class of Borel functions. In replacing limits by limsups character degradation occurs leading to functions higher up the analytical (projective) hierarchy. In this connection we have previously argued [9, § 5] that $\boldsymbol{\Delta}_2^1$ is a most attractive class of sets within which to carry out the analyses of RV. When drawing in the Beurling operation $\circ_\varphi$, this argument needed amplification – see [13, § 11]. Here, when extending the argument to 'essential limits', we again point to the further attractions of the *provably* $\mathbf{\Delta}^1_2$ class of § 8.2. Working with 'nice versions' of functions – removing a pathological set covered by a $\mathcal{G}_\delta$ set, as in the Appendix – via the Baire-Kuratowski or Luzin Continuity Theorems [54, Th. 8.1, 8.2], we remain in the realm of Baire/measurable sets, as though under the sway of LM or PB. This is because the Baire Category Theorem, BC, suffices here. Indeed, BC is equivalent to DC; see [16] and the literature cited there.

9.2 *Smallwood's theorem.* Essential (or approximate) limits go back to work of Denjoy in 1916 on approximate continuity, and Khintchine in 1924 and 1927. An early textbook treatment is in Saks [60, IX.10]. Smallwood's Theorem [62] reconciles the Denjoy and Khintchine approaches: for $E \subseteq \mathbb{R}$ Lebesgue-measurable, $f : \mathbb{R} \to \mathbb{R}$ measurable, $x_0 \in \mathbb{R}$, f has approximate limit L at x_0 (i.e. $\exists$ess- $\lim f(x_0) = L$) if and only if there exists a measurable set $F \subseteq E$ with density 1 at x_0 and

$$\lim_{x\in F, x\to x_0} f(x) = L.$$

Such matters are important in probability theory; see e.g. the survey of Geman and Horowitz [28, Appendix: Metric density and approximate limits, 22-24], and the lecture notes of Adler [2, IV.4.6].

For essential (or approximate) semi-continuity, see Zink [65].

9.3 *Croftian theory and admissible sequences.* Croft's theorem says that for a continuous function f, the existence of all the sequential limits (as $n \to \infty$) of $f(nh)$ for all $h > 0$ implies that of the continuous limit of $f(x)$. Kingman [41, 42] re-writes this additively, so working with $f(\log n + x)$, and generalizes the Croft setting to ask for conditions on c_n for a similar result to apply for sequential limits of $f(c_n + x)$. Roughly speaking, the condition needed for a croftian theorem to hold here is the Kingman condition (KC) of §2, namely

$$c_{n+1} - c_n \to 0$$

(compare our *admissible sequences*, in additive and multiplicative notations). As in the work of Kingman [41, 42] and Kendall [40], the key role of the Baire category theorem is clear in the following further generalization of Vinokurov [64]: for f as above and c_n satisfying Kingman's condition, the condition on the set E needed for the implication from

$$f(c_n + x) \to L \qquad (x \in E)$$

to

$$f(x) \to L$$

is that E be non-meagre. For further results of this type, see Fehér at al. [25]; cf. [61].

9.4 *Regularly varying measures.* For random vectors X in $\mathbb{R}^d$, a theory of regularly varying measures can be based on the definition (suggested by Kendall's Theorem)

$$n\mathbb{P}(X/a_n \in \ .) \to \mu(.) \qquad (n \to \infty)$$

for $a_n \nearrow \infty$, vague convergence, and suitably restricted μ. Then regular variation is present, as for some $\alpha > 0$

$$\mu(tA) \sim t^{\alpha}\mu(A)$$

as in Theorem 6 (and then a_n is regularly varying). See e.g. Hult and Linskog [30], Hult et al. [31]. This approach is Kendall-like, as it is entirely sequential. It can be extended to infinite-dimensional settings, and is widely used nowadays in probability (theory and applications).

9.5 *Thinning: Steinhaus-Weil aspects.* The two main ingredients in verifying that the Kendall criterion yields regular variation are: the croftian property of the set C in Prop. 2, and the Steinhaus-Weil property of the test set B – that BB^{-1} contains an interval J around 1. Recall that the latter guarantees that $C(s) = B \cap s^{-1}B$ is non-empty for $s \in J$, and so for $s \in J$ and $\lambda \in C(s)$

$$k_s(\lambda x_n) \to g(s\lambda)/g(\lambda), \text{ as } n \to \infty.$$

The Steinhaus-Weil property can hold for nowhere dense sets (cf. a multiplicative analogue of the classical Cantor excluded middle thirds); indeed there is a rich family of such sets – see the SW property used in [15].

However, Prop. 2 relies both on the Baire function g having a point of continuity in C, and on C having the property that, for an additively admissible sequence c_n, the tail union of its translates $\bigcup_{n \geq m}(c_n + C)$ contains quasi all of an infinite half-ray. Just as in Vinokurov's result (§ 9.3), C here cannot be negligible.

That said, we note that, taken together, Theorem 1 and 1M already imply (on taking $a_n := 1/f(x_n)$) the Characterization Theorem K of § 2 with the global hypothesis of convergence $f(tx)/f(t) \to g(t)$ 'for all t' much weakened to 'for all t on a *non-negligible set*'. On the other hand, it is known [15] that a further thinning is possible: to sets having the SW property locally. One is thus led to ask, given the convergence ess-$\lim_{x\to\infty} k_s(x)$ on an s-interval, whether the Theorem 1 can be further thinned, despite the possible negligibility of $C(s)$ itself.

9.6 *Functional equations, FE.* For an account of the literature of (GS) and related equations see [1, Ch. 19] and the more recent [18], cf. the summary in [51, § 1]. In our context it is natural to restrict solutions of (GS) and of the related Chudziak-Jabłońska equation

$$H(x \circ_\eta y) = H(x)H(y), \tag{CJ}$$

with η continuous, to be non-negative and locally bounded. It then emerges from [19] (cf. [53, § 9.5] for a more direct approach), [32, 33, 34, 35] and especially [34], that, provided the function H is *non-trivial* (i.e. its range is not a subset of $\{0,1\}$), then local boundedness of the solution H implies continuity. (Note the trivial counter-example: the Dirichlet function $H = \mathbf{1}_{\mathbb{Q}}$ for $\eta(t) = 1 + t$.) This observation includes the case of solutions of (GS) which take the form $\eta(t) = \eta_\rho(t)$ for some $\rho \geq 0$ and $t > -1/\rho$. The case $\rho = \infty$, corresponding to $x \circ_\eta y = xy$, is just another instance of (CFE).

By Theorem 6, taking g non-trivial in Theorems 4 and 4M ensures the corresponding K is likewise non-trivial and so continuous.

Matters are the same in the more general (BG) equation. The case for $\sigma(t) = 1 + st$, with $s \geq 0$ is typified by $s = 1$ via scaling (save for the case $s = 0$, reduced via logarithms to $s = 1$); then $u \circ_\sigma v = u + v + uv = (u+1)(v+1) - 1$ and here the (BG) equation reduces to

$$K(x \circ_\eta y) + 1 = (K(x) + 1)(K(y) + 1),$$

so that $H(x) := K(x) + 1$ is locally bounded and so continuous provided K (being non-negative) is non-zero – by Jabłońska's theorems in [34]. But here, again Theorem 6, since g is assumed non-zero in Theorems 5 and 5M, ensures the corresponding K is again continuous.

The continuous solutions of (BG) are given in the table below. (The FE literature also includes studies of the case where on the right $\circ_\sigma$ is replaced by a semigroup operation $\circ$ as in [20, 21].)

In the table, the four corner-formulas correspond to classical variants of the Cauchy functional equation (CFE). For completeness we include the proof; this proceeds by a straightforward reduction to a classical variant of (CFE) by an appropriate shift and rescaling, similar to the reduction from K to H above. The notation $\circ_r$ etc. below refers to the Popa operation $\circ_\eta$ with parameter r, i.e. the case $\eta = \eta_r$.

Proposition 3 ([52, Prop. A]; cf. [20]). *For $\circ_\eta = \circ_r, \circ_\sigma = \circ_s$, and K Baire/measurable satisfying (BG), there is $\kappa \in \mathbb{R}$ so that $K(t)$ is given by:*

Popa parameter	$s = 0$	$s \in (0, \infty)$	$s = \infty$
$r = 0$	κt	$(e^{\kappa t} - 1)/s$	$e^{\kappa t}$
$r \in (0, \infty)$	$\kappa \log(1 + rt)$	$[(1 + rt)^\kappa - 1]/s$	$(1 + rt)^\kappa$
$r = \infty$	$\kappa \log t$	$(t^\kappa - 1)/s$	t^κ

Proof. Each case reduces to (CFE) on $\mathbb{R}_+$, or a classical variant by an appropriate shift and rescaling. For instance, given K, for $r, s > 0$ set

$$F(t) := 1 + sK((t-1)/r) : \qquad f(\tau) = (K(1 + r\tau) - 1)/s.$$

Then with $u = 1 + rx, v = 1 + ry$, as $(uv - 1)/r = x \circ_r y$,

$$F(uv) = 1 + sK(x \circ_r y) = 1 + sK(x) + sK(y) + s^2K(x)K(y) = F(u)F(v),$$

for $u, v \geq 0$. So, as F is Baire/measurable (see again [46, § 13]), $F(t) = t^\gamma$ and so $K(t) = [(1 + rt)^\gamma - 1]/s$. The remaining cases are similar. $\square$

In the language of isomorphisms η_ρ, exp, log, we can rephrase the above more succinctly as follows:

Popa parameter	$\sigma = 0$	$\sigma \in (0, \infty)$	$\sigma = \infty$
$\rho = 0$	κt	$\eta_\sigma^{-1}(e^{\kappa t})$	$e^{\kappa t}$
$\rho \in (0, \infty)$	$\log \eta_\rho(t)^\kappa$	$\eta_\sigma^{-1}(\eta_\rho(t)^\kappa)$	$\eta_\rho(t)^\kappa$
$\rho = \infty$	$\log t^\kappa$	$\eta_\sigma^{-1}(t^\kappa)$	t^κ

9.7 *Open Question.* In passing, motivated by the context of Prop. 3, we leave open the question whether Prop. 1 (based on an affine action between two sets A, B) has an analogue for more general continuous $h(z, s)$ in the spirit of the 'Miller homotopies' [48], cf. [11]. (We have in mind something along the lines: for convergent sequences $z_n \to z_0$, for almost all b near b_0 there are infinitely many m with $H(b, z_m) \in A$ – here with H the appropriate local inverse at $b_0 = h(z_0, a_0)$.) See [29] as to a possible approach for replacing differentiability by 'Radon-Nikodym differentiability' as in the 'Functionwise Steinhaus-Weil Theorem', wherein $f(A \times B)$ has an interior point (originating with Marcin Kuczma [44]).

Appendix: Relevant aspects of coding.

Theorem 7 above relied on the ability to refer to various subsets of the real line, especially open sets, in terms of 'codes'. Our canonical sources there were [39, Ch.V] on the analytical hierarchy (and the note [39, V.40B] on classical versus effective descriptive set theory), and our recent survey [16], and for coding the wide-ranging use in [63, II.1.1, 25-33] and the much more minimal amount in [27, § 2, p. 93]. Here we give some examples to help clarify the full effect on the *analytical quantifier blocks* in Theorem 7. We begin with some notation.

Let $\{I_n\}_{n\in\mathbb{N}}$ enumerate (constructively) all the rational-ended intervals, with $I_n = (l_n, r_n)$. Write $\mathbb{M}$ for the odd natural numbers; for $a \subseteq \mathbb{N}$ we may extract an n^{th} canonical subset of a and also an open set naturally 'coded' by a by setting:

$$a(n) = a \cap \{2^n m : m \in \mathbb{M}\}, \qquad G(a) := \bigcup\nolimits_{n\in a} I_n.$$

We identify $a \subseteq \mathbb{N}$ with the real number in $\{0,1\}^{\mathbb{N}}$ whose binary expansion is the indicator function of a. Thus $\{a : m \in a\}$ is open (being the set of reals with m-th binary digit $=1$).

The following are examples of Borel sets, using semi-formal predicates:

$$\begin{aligned}
&\{a \;:\; G(a) \text{ is unbounded}\} = \\
&\{a \;:\; (\forall k \in \mathbb{N})(\exists q \in \mathbb{Q})(\exists m \in \mathbb{N})[(q > k) \;\&\; m \in a \;\&\; q \in I_m]\}; \\
&\{a \;:\; G(a) \text{ is everywhere dense}\} = \\
&\{a \;:\; (\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(\exists q \in \mathbb{Q})[m \in a \;\&\; q \in I_n \;\&\; q \in I_m]\}.
\end{aligned}$$

The defining statements here are said to be *arithmetic* since the quantifiers are 'arithmetic': they all range over the (countable) set of natural or rational numbers, and the 'matrix' (the expression in square brackets – not containing quantifiers) is built from elementary relations like $l_n < q < r_n$ and $m \in a$, and these may be viewed as (codes for the very simple, basic) open sets containing the real numbers a. The two examples above are Borel, as they may be constructed using countable unions and intersections (corresponding to the arithmetic quantifiers) from basic open sets; for example, the second set is $\mathcal{G}_\delta$, since it has the form

$$\bigcap\nolimits_{n\in\mathbb{N}} \bigcup\nolimits_{m\in\mathbb{N}, q\in\mathbb{Q}} \{a : m \in a \;\&\; q \in I_m \;\&\; q \in I_n\}.$$

For f a Baire function and L fixed, the set

$$H_m(f) := \{x : |f(x) - L| < 1/m\}$$

is Baire. If f happened to be continuous, this set would be open, and so coded as $G(a_m)$ with $a_m := \{n : I_n \subseteq H_m(f)\}$, i.e. by a set of natural numbers, or, equivalently, by a single real number. For a general Baire f, since we are prepared to neglect meagre sets, as suggested in § 2 Remark 2, we can make a *simplifying assumption*: regard H_m as coded by some open set, $G(b_m)$ say, less a union of closed nowhere dense sets – in essence use a nice version of f; passing to the sequence of the complements of the closed nowhere dense sets, we view the removal of their union as an intersection of open sets, the n^{th} one coded by the subset $c_m(n)$ of c_m, say. So, for example, with our simplifying assumption, $H_m(f)$ may be regarded as being the $\mathcal{G}_\delta$ set

$$G(b_m) \cap \bigcap\nolimits_{n\in\mathbb{N}} G(c_m(n)).$$

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