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Angular measures and Birkhoff orthogonality in Minkowski planes

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Abstract. Let x and y be two unit vectors in a normed plane $\mathbb{R}^2$. We say that x is *Birkhoff orthogonal* to y if the line through x in the direction y supports the unit disc. A *B-measure* (Fankhänel in Beitr Algebra Geom 52(2):335–342, 2011) is an angular measure μ on the unit circle for which $\mu(C) = \pi/2$ whenever C is a shorter arc of the unit circle connecting two Birkhoff orthogonal points. We present a characterization of the normed planes that admit a B-measure.

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1. Introduction

Let K be an origin-symmetric *convex body* in the plane, that is, a compact convex set with non-empty interior in $\mathbb{R}^2$, and consider the normed plane $(\mathbb{R}^2, \|\cdot\|_K)$, where $\|x\|_K = \min\{\lambda > 0 \,:\, x \in \lambda K\}$ for any $x \in \mathbb{R}^2$. Then K is the *unit ball* of the norm, and its boundary $\operatorname{bd} K$ the *unit circle*.

Let $x, y \in \operatorname{bd} K$ be two unit vectors in $\mathbb{R}^2$. We say that x is *Birkhoff orthogonal* to y, and denote it by $x \dashv y$, if $\|x\|_K \le \|x + ty\|_K$ for all $t \in \mathbb{R}$. Geometrically, this means that the line through the point x in the direction y supports the unit ball K. In general, Birkhoff orthogonality is not a symmetric relation. Normed planes where Birkhoff orthogonality is symmetric are called

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Radon planes and the boundaries of their unit balls *Radon curves* (see the survey [5]).

A Borel measure μ on $\operatorname{bd} K$ is called an *angular measure*, if $\mu(\operatorname{bd} K) = 2\pi$, $\mu(X) = \mu(-X)$ for every Borel subset X of $\operatorname{bd} K$, and μ is *continuous*, that is, $\mu(\{x\}) = 0$ for every $x \in \operatorname{bd} K$. There always exists an angular measure on $\operatorname{bd} K$, such as the one-dimensional Hausdorff measure on $\operatorname{bd} K$ normalized to 2π, but an arbitrary angular measure does not necessarily have any relation to the geometry of $(\mathbb{R}^2, \|\cdot\|_K)$. A natural problem then is to find angular measures with interesting geometric properties. For instance, Brass [2] showed that whenever the unit ball is not a parallelogram, there is an angular measure in which the angles of any equilateral triangle are equal. This type of angular measure is very useful in studying packings of unit balls [2,8]. Angular measures with other properties have been proposed; see the survey [1, Section 4] for an overview. An angular measure μ is called a *B-measure* [3] if $\mu(C) = \pi/2$ for every closed arc C of $\operatorname{bd} K$ that contains no opposite points of $\operatorname{bd} K$, and whose endpoints x and y satisfy $x \dashv y$.

The main result of this note (Theorem 1) is a characterization of the normed planes $(\mathbb{R}^2, \|\cdot\|_K)$ which admit a B-measure. In order to formulate this theorem, we need to introduce two subsets of $\operatorname{bd} K$.

We call a point x in $\operatorname{bd} K$ an *Auerbach point*, if there is a $y \in \operatorname{bd} K$ such that $x \dashv y$ and $y \dashv x$. In this case we say that x and y form an *Auerbach pair*. It is well known that Auerbach points exist for any norm [9, Section 3.2]. We denote the set of Auerbach points of K by $A(K)$. Note that $A(K)$ is a closed subset of $\operatorname{bd} K$. We denote the union of open non-degenerate line segments contained in $\operatorname{bd} K$ by $E(K)$.

Theorem 1. *Let K be an origin-symmetric convex body in $\mathbb{R}^2$. Then there is a B-measure on* $\operatorname{bd} K$ *if, and only if, the set $A(K) \setminus E(K)$ is uncountable.*

This is a strengthening of a result of Fankhänel [3, Theorem 1], where the existence of a B-measure is shown under the condition that $A(K) \setminus E(K)$ contains an arc. (Fankhänel does not explicitly exclude line segments, but it is clear that they have to be excluded, as line segments in $A(K)$ necessarily have measure 0 for any B-measure; see Lemma 3.) We prove Theorem 1 in Section 2, where we also present a smooth, strictly convex, centrally symmetric planar body K such that $A(K)$ is the union of two disjoint copies of the Cantor set and a countable set of isolated points (Example 4). Thus, $A(K)$ is of Lebesgue measure zero and yet, by Theorem 1, there is a B-measure on $\operatorname{bd} K$.

We recall that a subset of a topological space is called *perfect* if it is closed and has no isolated point. Recall that the *support* $\operatorname{supp}(\mu)$ of a Borel measure μ on a topological space X is the set of all $x \in X$ such that all open sets containing x have positive μ-measure. It is easy to see that the support of any continuous measure is a perfect set. In the proof of Theorem 1, we rely on the following converse for $X = [0, 1]$.

Proposition 2. *Let $H \subset [0,1]$ be a non-empty, perfect set. Then there is a continuous probability measure on $[0,1]$ whose support is H.*

This is a well-known result holding more generally for any separable complete metric space [6, Chapter II, Theorem 8.1], but for the convenience of the reader we present an explicit construction for this special case in Section 3. It is well known that every non-empty perfect set is uncountable [7, Theorem 2.43] and every uncountable Borel set contains a perfect set [4, Section 6B]. (There is an even larger class, the analytic sets, with this property [4], but we will only need it for F_σ sets).

2. The Auerbach set and B-measure

Given two non-opposite points $a, b \in \operatorname{bd} K$, we denote by $\sphericalangle(a,b)$ the closed arc from a to b that does not contain any opposite pairs of points. We denote the closed line segment with endpoints $a, b \in \mathbb{R}^2$ by $[a,b]$.

Lemma 3. *Let K be an origin-symmetric convex body in $\mathbb{R}^2$ and μ be a B-measure on $\operatorname{bd} K$. Then $\operatorname{supp}(\mu) \subseteq A(K)\backslash E(K)$.*

Proof. Let $x \in E(K)$. Then $x \in [x^-, x^+] \subset \operatorname{bd} K$ for some x^-, x^+ with x, x^-, x^+ distinct. Let $y \in \operatorname{bd} K$ be parallel to $[x^-, x^+]$. Since $x^-, x^+ \dashv y$, we have $\mu([x^+, y]) = \mu([x^-, y]) = \pi/2$, hence $\mu([x^-, x^+]) = 0$ and $x \notin \operatorname{supp}(\mu)$.

Next, let $x \in \operatorname{bd} K \backslash A(K)$. Let $y_1, y_2 \in \operatorname{bd} K$ such that $x \dashv y_1$ and $y_2 \dashv x$. Then $y_1 \neq y_2$. By possibly replacing y_2 by $-y_2$, we assume without loss of generality that y_1 and y_2 are in the same open half plane bounded by the line ox. By possibly replacing x by $-x$, we may also assume without loss of generality that y_2 and x are in the same open half plane bounded by oy_1. Let x_1 and x_2 be points on the same side of oy_1 as x such that $y_1 \dashv x_1$ and $x_2 \dashv y_2$. Then $x_1, x_2 \neq x$. Because y_2 is between x and y_1, we have that x_1 and x_2 are in opposite open half planes bounded by ox. As above, since μ is a B-measure, $\mu(\sphericalangle(x_1, x_2)) = \mu(\sphericalangle(x, x_1)) = \mu(\sphericalangle(x, x_2)) = 0$, hence $x \notin \operatorname{supp}(\mu)$. □

Proof of Theorem 1. Let μ be a B-measure on $\operatorname{bd} K$. Then $\operatorname{supp}(\mu)$ is a perfect set, hence uncountable, and Lemma 3 gives that $A(K)\backslash E(K)$ is uncountable.

Conversely, assume that $\widetilde{A} := A(K)\backslash E(K)$ is uncountable. We next find an appropriate perfect subset of $\widetilde{A}$ and use Proposition 2 to define a B-measure on $\operatorname{bd} K$. We first need to define an auxiliary map $\phi : \widetilde{A} \to A(K)$ by setting $\phi(x)$ to be the first $y \in A(K)$ in the positive direction along $\operatorname{bd} K$ from x so that $x \dashv y$ and $y \dashv x$. Then ϕ is monotone, but not necessarily injective. However, if $\phi(x_1) = \phi(x_2)$, then $x_1 \dashv y$ and $x_2 \dashv y$, as well as x_1 and x_2 being on the same side of line oy. Thus $[x_1, x_2]$ is a line segment on $\operatorname{bd} K$. Since the set

$$E'(K) := \{y \in \operatorname{bd} K \ : \ K \text{ has more than one supporting line at } y\}$$

is countable, it follows that for any given $y \in A(K)$, there are at most two values of $x \in \widetilde{A}$ such that $\phi(x) = y$, and there are at most countably many $y \in A(K)$ for which there is more than one $x \in \widetilde{A}$ such that $\phi(x) = y$. In particular, ϕ is a Borel measurable map.

We next find an appropriate arc $\sphericalangle(a,b)$ such that $\sphericalangle(a,b) \cap \widetilde{A}$ is uncountable. For any $x \in \operatorname{bd} K$, let x^+ denote the first element of $\widetilde{A}$ in the positive direction from x, and let x^- be the first element of $\widetilde{A}$ in the negative direction from x. (If $x \in \widetilde{A}$ then $x = x^- = x^+$).

Let $\overline{E}(K)$ denote the union of the *closed* line segments on $\operatorname{bd} K$. Then $\overline{E}(K)$ is the union of $E(K)$ with a countable set. Observe that for any $p \in \operatorname{bd} K$, the set $\phi^{-1}(p)$ contains at most two points. Thus, $\phi^{-1}(E'(K))$ is countable. Moreover, $\phi^{-1}(\bar{E}(K))$ is countable, since ϕ takes at most one value on an open line segment on $\operatorname{bd}(K)$. Fix an element

$$a \in A(K) \setminus \left[\overline{E}(K) \cup E'(K) \cup \phi^{-1}\left(\bar{E}(K) \cup E'(K)\right)\right],$$

and let $b = \phi(a)$. Since $a \notin E'(K)$, the only two points of $\operatorname{bd}(K)$ that form an Auerbach pair with a are $\pm b$. Since $a \notin \overline{E}(K)$, the only two points of $\operatorname{bd}(K)$ that form an Auerbach pair with b are $\pm a$. Since $a \notin \phi^{-1}(E(K))$, we have $b \in A(K) \setminus E(K)$. It follows that $\phi(b) = -a, \phi(-a) = -b$ and $\phi(-b) = a$.

We also obtain that $\sphericalangle(a,b) \cap \widetilde{A}$ or $\sphericalangle(b,-a) \cap \widetilde{A}$ is uncountable. Thus we may assume without loss of generality that $\sphericalangle(a,b) \cap \widetilde{A}$ is uncountable, where $\phi(a) = b$ and $\phi(b) = -a$, so it contains a perfect set, and by Proposition 2 there is a continuous probability measure ν on the Borel sets of $\operatorname{bd} K$ with $\operatorname{supp}(\nu) \subseteq \sphericalangle(a,b) \cap \widetilde{A}$. We use ν to define the B-measure as follows. For any Borel set $S \subseteq \operatorname{bd} K$, let

$$\mu(S) := \frac{\pi}{2}\left[\nu(S) + \nu(-S) + \nu(\phi^{-1}(S)) + \nu(\phi^{-1}(-S))\right]. \tag{1}$$

Then μ is clearly an angular measure. Showing that μ is a B-measure is somewhat technical, mainly because $\dashv$ is not in general a symmetric relation. Let $x, y \in \operatorname{bd} K$ with $x \dashv y$. We have to show that $\mu(\sphericalangle(x,y)) = \pi/2$. After possibly replacing x by $-x$ and y by $-y$, we may assume that $x \in \sphericalangle(a,b) \cup \sphericalangle(b,-a)$ and $y \in \sphericalangle(a,b) \cup \sphericalangle(b,-a)$.

Case 1: $x \in \sphericalangle(a,b)$. Then either $y \in \sphericalangle(a,b)$ or $y \in \sphericalangle(b,-a) \setminus \{b\}$.

Case 1.1: $y \in \sphericalangle(a,b)$. There are two cases depending on the relative position of x and y.

Case 1.1.1: $x \in \sphericalangle(a,y)$. Since $a \notin \overline{E}(K)$, we obtain $x = a$, and since $a \notin E'(K)$, we obtain $y = b$. Hence, $\mu(\sphericalangle(x,y)) = \pi/2$ as required.

Case 1.1.2: $x \in \sphericalangle(y,b)$. Since $b \notin E'(K)$, we obtain $y = a$, and since $b \notin \overline{E}(K)$, we obtain $x = b$, and again $\mu(\sphericalangle(x,y)) = \pi/2$.

Case 1.2: $y \in \sphericalangle(b,-a) \setminus \{b\}$. In order to show that $\mu(\sphericalangle(x,y)) = \pi/2$, it will be sufficient to show that $\phi^{-1}(\sphericalangle(b,y))$ equals $\sphericalangle(a,x) \cap \widetilde{A}$ up to ν-measure 0. In fact, we show that

$$\begin{aligned}&\phi^{-1}(\sphericalangle(b,y^+))\ \cup\ (\{x\}\cap\widetilde{A})\cup\phi^{-1}(E(K))\\=&(\sphericalangle(a,x)\cap\widetilde{A})\ \cup\ \phi^{-1}(\{b,y^+\})\cup\phi^{-1}(E(K)).\end{aligned} \tag{2}$$

First, let $p\in\phi^{-1}(\sphericalangle(b,y^+))\backslash\phi^{-1}(E(K))$. Then $\phi(p)\in\sphericalangle(b,y^+)$ and $p\in\widetilde{A}$. Without loss of generality, $\phi(p)\neq b,y^+$, and we want to show that $p\in\sphericalangle(a,x)$. Clearly, $p\in\sphericalangle(a,b)$. Suppose that $p\in\sphericalangle(x,b)$ and $p\neq x$. It follows from $p\dashv\phi(p)$ and $x\dashv y$ that $\phi(p)\notin\sphericalangle(b,y)\backslash\{y\}$, since otherwise $p=x$. Therefore, $\phi(p)\in\sphericalangle(y,y^+)$. However, since $\phi(p),y^+\in\widetilde{A}$, we obtain the contradiction $\phi(p)=y^+$. Therefore, $p\notin\sphericalangle(x,b)\backslash\{x\}$, and it follows that $p\in\sphericalangle(a,x)$, which finishes the proof of the $\subseteq$-inclusion of (2).

For the opposite inclusion, we assume without loss of generality that $p\in\sphericalangle(a,x)\cap\widetilde{A}$ and $\phi(p)\neq b,y^+$. Suppose that $\phi(p)\notin\sphericalangle(b,y^+)$. Then $y^+\in\sphericalangle(b,\phi(p))\backslash\{\phi(p)\}$. By considering $p\dashv\phi(p)$ and $x\dashv y$, we obtain that $p=x$, so $p\in\{x\}\cap\widetilde{A}$. This proves the $\supseteq$-inclusion of (2).

Case 2: $x\in\sphericalangle(b,-a)$. This case is very similar to Case 1 and we only summarize the argument.

Case 2.1: $y\in\sphericalangle(b,-a)$. As in Case 1.1, we use $a,b\notin E'(K)\cup\overline{E}(K)$ to obtain that $\{x,y\}=\{a,b\}$.

Case 2.2: $y\in\sphericalangle(a,b)$. In an almost identical way as in Case 1.2, we can show that

$$\begin{aligned}&\phi^{-1}(\sphericalangle(b,x^+))\ \cup\ (\{y\}\cap\widetilde{A})\cup\phi^{-1}(E(K))\\=&\ (\sphericalangle(a,y)\cap\widetilde{A})\ \cup\ \phi^{-1}(\{b,x^+\})\cup\phi^{-1}(E(K)),\end{aligned}$$

from which it follows that $\nu(\sphericalangle(b,x))=\nu(\sphericalangle(a,y))$, hence $\mu(\sphericalangle(x,y)=\pi/2$ by (1).

This completes the proof of Theorem 1. ☐

Example 4. We present a smooth, strictly convex, origin-symmetric planar body K such that $A(K)$ is the union of two disjoint copies of the Cantor set and a countable set of isolated points.

First, let D denote the Euclidean unit disk centered at the origin, and let C be the shorter arc connecting the two points whose angles with the positive x axis are $-\pi/4$ and $\pi/4$. Let C_0 denote the Cantor set in C. Now, C_0 can be written as

$$C_0=C\backslash\bigcup_{n=1}^{\infty}I_n,$$

where the I_n are disjoint open arcs in C.

For each $n\in\mathbb{Z}^+$, we construct a smooth and strictly convex curve C_n connecting the two endpoints of I_n with the following properties.

1. C_n has the same tangents at the endpoints as D;
2. C_n is contained in $\operatorname{conv} I_n$;

3. For any point x of C_n, the tangent of C_n at x is orthogonal (in the Euclidean sense) to x if, and only if, x is the midpoint or an endpoint of C_n.

Consider the bump function

$$\Psi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } x \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that Ψ is non-negative, smooth, its support is $[-1,1]$, and the only points in its support where the derivative is zero are $-1, 1$ and $1/2$.

Let the endpoints of I_n be $(\cos\alpha_n, \sin\alpha_n)$ and $(\cos\beta_n, \sin\beta_n)$, where $\alpha_n < \beta_n$. Let C_n be the curve

$$\varphi \mapsto \left(1 - \varepsilon\Psi\left(\frac{2}{\beta_n - \alpha_n}\left[\varphi - \frac{\alpha_n+\beta_n}{2}\right]\right)\right)(\cos\varphi, \sin\varphi), \quad \varphi \in [\alpha_n, \beta_n],$$

for some small $\varepsilon > 0$.

Clearly, C_n is a smooth curve, and if ε is sufficiently small, then it is also strictly convex. Moreover, C_n satisfies Property 1, as $\Psi'(-1) = \Psi'(1) = 0$. If ε is sufficiently small, then C_n satisfies Property 2 as well. Finally, to verify Property 3, observe that the tangent of C_n is orthogonal to $(\cos\varphi, \sin\varphi) \in C_n$ if, and only if, the derivative of

$$\varphi \mapsto 1 - \varepsilon\Psi\left(\frac{2}{\beta_n - \alpha_n}\left[\varphi - \frac{\alpha_n+\beta_n}{2}\right]\right)$$

vanishes at φ. However, this is only the case at the midpoint and two endpoints of C_n.

The closed curve

$$L := (\operatorname{bd} D \setminus (C \cup -C)) \cup (C_0 \cup -C_0) \cup \bigcup_{n=1}^{\infty} (C_n \cup -C_n)$$

is the boundary of a smooth, strictly convex, origin-symmetric planar body K, say. In order to identify the Auerbach points of K, first observe that if $x, y \in L$ form an Auerbach pair in K, then x and y are orthogonal in the Euclidean sense. (The converse does not hold, of course.) By this observation and Property 3, for each $n \in \mathbb{Z}^+$, the only Auerbach point in the relative interior of the arc C_n is the midpoint of C_n. The same holds for $-C_n$. Again by the observation, all points of $C_0 \cup -C_0$ are Auerbach points. Finally, again by the observation, the set of Auerbach points of $(\operatorname{bd} D \setminus (C \cup -C))$ is the rotation of the previously described set of Auerbach points in $(C_0 \cup -C_0) \cup \bigcup_{n=1}^{\infty} (C_n \cup -C_n)$ by an angle of $\pi/2$. It follows that $A(K)$ is the union of two disjoint copies of the Cantor set and a countable set of isolated points. $\square$

3. Proof of Proposition 2

We may assume that $0, 1 \in H$. Enumerate the components of $\mathbb{R}\backslash H$ as $I_0, I_1, \ldots$, where $I_0 := (-\infty, 0)$ and $I_1 := (1, \infty)$. We will recursively assign a real number y_n to each open interval I_n. Let $y_0 := 0$ and $y_1 := 1$.

If y_k has already been defined for all $k < n$, let

$$y_n := \frac{1}{2}\left(\max_{\substack{\ell<n\\ I_\ell<I_n}} y_\ell + \min_{\substack{\ell<n\\ I_\ell>I_n}} y_\ell\right),$$

that is, we consider the two intervals with indices less than n just below and just above I_n, and y_n is the average of the two values assigned to these two intervals.

We define a function f on $\mathbb{R}$ as follows. First, on $\mathbb{R}\backslash H$, let $f|_{I_n} = y_n$. To extend f to $\mathbb{R}$, we set

$$a_x := \sup(-\infty, x)\backslash H, \quad \text{and} \quad b_x := \inf(x, \infty)\backslash H. \tag{3}$$

If $x \in H$ and $a_x = b_x$, then the left limit, $f(a_x-)$, of f at a_x clearly equals the right limit $f(b_x+)$. Thus, the function

$$f(x) := \begin{cases} y_n & \text{if } x \in I_n; \\ f(a_x-) = f(b_x+) & \text{if } x \in H \text{ and } x = a_x = b_x; \\ f(a_x-)\dfrac{b_x - x}{b_x - a_x} + f(b_x+)\dfrac{x - a_x}{b_x - a_x} & \text{if } x \in H \text{ and } a_x < b_x \end{cases}$$

is continuous, strictly increasing on H, and locally constant on $\mathbb{R}\backslash H$.

Finally, let μ_0 denote the Lebesgue-Stieltjes measure corresponding to f, and μ_1 the measure $\mu_1(A) = \lambda(A \cap H)$, where λ is Lebesgue measure. Then $\mu = \mu_0 + \mu_1$ is a continuous measure, and $\operatorname{supp}\mu \subseteq H$.

To show the reverse inclusion, let I be an open interval and assume that $I \cap H \neq \emptyset$. If $I \cap H$ is of positive Lebesgue measure, then $\mu(I) > \mu_1(I) > 0$. Otherwise, I is intersected by at least two I_k. Indeed, if only one I_k intersected I, then $I \cap H$ would be the union of at most two intervals, contradicting that H is perfect and of Lebesgue measure zero.

Since the values of f on distinct intervals I_k are distinct, f is not constant on I, and hence, $\mu(I) > \mu_0(I) > 0$, completing the proof of Proposition 2.

The total measure is $\mu(\mathbb{R}) = \mu_0(\mathbb{R}) + \mu_1(\mathbb{R}) = 1 + \lambda(H) \in [1, 2]$, and thus $\nu = \mu/\mu(\mathbb{R})$ is a probability measure with the desired properties. □

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