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Regarding 'Leibniz Equivalence'

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Abstract
Leibniz Equivalence is a principle of applied mathematics that is widely assumed in both general relativity textbooks and in the philosophical literature on Einstein's hole argument. In this article, I clarify an ambiguity in the statement of this Leibniz Equivalence, and argue that the relevant expression of it for the hole argument is strictly false. I then show that the hole argument still succeeds as a refutation of manifold substantivalism; however, recent proposals that the hole argument is undermined by principles of representational equivalence do not fare so well.

Keywords General relativity · Spacetime · Substantivalism · Relationism · Leibniz equivalence · Hole argument

1 Introduction

When using mathematical language to represent the physical world, it is tempting to develop principles for how to do it well. One such principle is what philosophers of physics call 'Leibniz Equivalence', which gives the plausible advice that, *if two mathematical structures are isomorphic, then they can both be used to accurately represent the same physical situation*.

Earman and Norton [6] formulated and named Leibniz Equivalence in the analysis of spacetime substantivalism and Einstein's hole argument. Since then Leibniz Equivalence has received a great deal of support, especially when the notion of isomorphism is given by the 'diffeomorphism freedom' of general relativity. For example, Brighouse writes that relationists about spacetime, "will hold that Leibniz equivalence is sufficient for physical equivalence" [3, p. 118]. Saunders gives a statement of Leibniz Equivalence very similar to the one above, and argues that the thesis "applies equally to any symmetry of a physical theory, when applied to the world as a whole, and to any transformation that can be only intrinsically defined"

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[26, p. 301]. Pooley describes "a family of responses to the spacetime version of the hole argument, namely those which, one way or another, claim that all diffeomorphic spacetime models represent exactly the same physical state of affairs, a position well-known in the literature as Leibniz Equivalence" [20, p. 375]. More recently, Weatherall claims that, "the moral of the hole argument is supposed to be that one must accept that diffeomorphic models represent the same physical situation" [27, p. 343]. And there are many more comments like this.

In this article I will examine this seemingly plausible principle more closely—and show a subtle sense in which it is false. Philosophers of physics will recognise this conclusion as potentially relevant to the spacetime substantivalism debate, where Leibniz Equivalence has been purported to do real philosophical work. I will argue on the contrary that Leibniz Equivalence is not so relevant to the substantivalism debate after all.

I begin in Sect. 2 by clarifying an ambiguity between two interpretations of Leibniz Equivalence, at least as it was originally baptised in Earman and Norton's analysis of spacetime substantivalism and the hole argument. Only one of them is actually relevant for the substantivalism debate, and it is this principle that I will argue is strictly false in Sect. 3. In Sect. 4 I show that this result turns out to have little effect on Einstein's hole argument. Rather, it shows that the hole argument is not so much a *reductio* of Leibniz Equivalence denial, as it is of manifold substantivalism in a more direct sense. Section 5 gives a critical discussion of recent work by Weatherall [27] suggesting that the hole argument depends on a misleading use mathematics. Section 6 is the conclusion.

2 Two Forms of Leibniz Equivalence

2.1 Weak and Strong Leibniz Equivalence

In a famous challenge to Clarke, Leibniz wrote that it would make no difference to transform a description of the universe by "changing east into west".[1] He took this to bode poorly for the spacetime substantivalism of Newton and Clarke: if the only difference to be found is in "the chimerical supposition of the reality of space in itself", then all the worse for the reality of space!

Leibniz's challenge can be formulated a little more powerfully in terms of the principle known as Leibniz Equivalence. The rough idea is that, insofar as spatially 'flipped' mathematical descriptions are isomorphic, Leibniz Equivalence implies that both provide equally accurate ways to represent the physical universe. If spacetime were a real entity, then there would seemingly be a distinction between these two descriptions. Leibniz Equivalence thus seems to provide a reason to reject the reality of spacetime. Can this argument be made in the more rigorous language of modern physics? The answer depends on how one interprets an ambiguity in the expression of Leibniz Equivalence.

[1] See Leibniz's third letter to Clarke, §5 [16, p. 14].

Let me begin with a somewhat informal discussion of two things 'Leibniz Equivalence' could mean, deferring a more precise technical discussion in spacetime physics for the next subsection. When one states that isomorphic structures can both be used to represent a physical situation accurately, does that mean that they can do so 'at once', in the sense of relating to the world through the same concrete interpretation? Or not necessarily?[2] The former might be said to express that two structures have 'co-representational capacity', whereas the latter expresses that they have 'equal representational capacity'[3]. Since the latter is a logically weaker statement, I'll refer to it as 'Weak' Leibniz Equivalence.

Definition 1 (*Weak Leibniz Equivalence*) Isomorphic mathematical structures can each be used with equal accuracy to represent a given physical situation (though not necessarily at once).

Example New Yorkers usually describe a set of locations on one side of Manhattan as 'East' and those on the other side as 'West' ('Not East'). But we could describe the same locations with those labels reversed. The two descriptions are isomorphic, and each one provides an accurate way to represent locations in Manhattan, just as Weak Leibniz Equivalence would have it. Of course, we cannot consistently use both in the same concrete interpretation without generating contradictions of multiple denotation, like: 'The New York Public Library is located on the East side and on the West side (not on the East side)'.[4] But, Weak Leibniz Equivalence is still satisfied, because it does not require the two descriptions to represent the same thing at once.

Regardless of its plausibility, Weak Leibniz Equivalence provides little help in formulating Leibniz's challenge to Clarke. A description of the universe, like a description of Manhattan, can be reversed East-to-West. But this does not prevent one from being a substantivalist, who would just view this as a relabelling of the real spacetime events! That the two isomorphic descriptions 'can' both be used to accurately represent spacetime does not by itself contradict the reality in spacetime.

In contrast, a logically stronger reading of Leibniz Equivalence may do some work for Leibniz's argument:

Definition 2 (*Strong Leibniz Equivalence*) Isomorphic mathematical structures can all be used with equal accuracy to represent a given physical situation, at once.

[2] This ambiguity is briefly discussed by Gryb and Thébault [10] in reference to Weatherall [27] and in an earlier version of this paper [22].

[3] The concept of a 'representational capacity' was applied to human cognition at least as early as Kant's *Critique of Cure Reason* (B72), and has been recently adopted by philosophers of scientific representation [c.f. 28, 30]. It was helpfully introduced into the hole argument literature by Weatherall [27].

[4] Goodman [9, §II,3] interprets art as capable of multiple denotation, and Priest [21] has analysed it in the context of paraconsistent logic. I am not aware of any response to Leibniz's challenge on these grounds.

If one adopts Strong Leibniz Equivalence, then isomorphic descriptions of Manhattan, spacetime, or anything else can be used to equally represent a given physical situation at once, in the same concrete interpretation. So, what of the paradoxes of multiple denotation? In both the Manhattan example and Leibniz's description of the universe, there are two ways to avoid the problem:

1. *Deny that the two descriptions are isomorphic:* one or the other might be accurate, but they are not isomorphic to each other; or
2. *Deny that either description is accurate:* neither accurately captures the facts.

In the Manhattan example, one might want to deny the isomorphism. For example, the sentence, 'Cars drive on the East side of a North-South street' is correct on one labelling of East-West and not on the other. But, if two descriptions really are isomorphic, then Strong Leibniz Equivalence implies that neither accurately captures the facts. This is the conclusion of the Leibniz argument about spacetime: statements about location in spacetime that are independent of the material it contains cannot be accurate. Reject such statements and the contradictions of multiple denotation go away.

One could go back and forth for some time about how to react to these informal examples. Things get more interesting when we eliminate those ambiguities and adopt a more mathematically practice language. So, let us now turn to Leibniz Equivalence in the context of modern spacetime physics.

2.2 Leibniz Equivalence in Spacetime Physics

Leibniz Equivalence was originally defined as a statement about representation in the context of general relativity. So, I will begin with a brief discussion of what it means for a model of general relativity to represent the world, through what Curiel [5] has called a 'concrete interpretation'. I then turn to clarifying the definition of Leibniz Equivalence in spacetime physics.

2.2.1 Concrete Interpretations

A model of general relativity is a linguistic structure that one can use to represent the world. I will assume this structure has the form of a *Lorentzian manifold*: each model is a pair (M, g_{ab}), where M is a smooth, connected, 4-dimensional manifold and g_{ab} is a smooth metric field of Lorentz signature (1, 3). Matter-energy is represented by an additional stress-energy field T_{ab}, but this can be uniquely reconstructed from g_{ab} using Einstein's equations. Other test-fields (which contribute negligibly to stress-energy) may be separately included as well. But for this discussion, no generality is lost in taking a model to be a simple pair (M, g_{ab}). Lorentzian manifolds come with a built-in standard of isomorphism, known as an 'isometry': we say that (M, g_{ab}) and $(\tilde{M}, \tilde{g}_{ab})$ are *isometric* if and only if there exists a diffeomorphism $\psi : M \rightarrow \tilde{M}$ such that the pullback of ψ preserves metric, in that $\psi^* g_{ab} = \tilde{g}_{ab}$.

When a Lorentzian manifold (M, g_{ab}) is used to represent the world in a concrete way, Curiel [5, §2] calls it a *concrete interpretation* in the spirit of model theory. That is, an interpretation can give truth to a sentence like, 'He is killing all of them.' For example, Hodges [13] charmingly gave that sentence the interpretation, "that 'he' is Alfonso Arblaster of 35 The Crescent, Beetleford, and that 'them' are the pigeons in his loft". Less violently, in general relativity, Curiel refers to them as "e.g., the fixation of a Tarskian family of models, or, less formally, the contents of a good, comprehensive text-book" [5]. For example, an interpretation can give (approximate) truth to a sentence like, 'The galaxy contains a Schwarzschild black hole (M, g_{ab}) of roughly 5×10^6 solar masses'. To make this sentence true, let (M, g_{ab}) be concretely interpreted to represent the region around the black hole at the centre of our galaxy. This is of course not the only concrete interpretation[5] of the (Schwarzschild) Lorentzian manifold (M, g_{ab}). For example: the same linguistic structure could be used to represent the (comparably large) black hole at the centre of Messier 61 with reasonable accuracy. For the purposes of this article, what is most relevant about the example is that we cannot use (M, g_{ab}) to represent both the centre of our galaxy and Messier 61 'at once'—meaning, 'in the same concrete interpretation'—on pain of contradictions of multiple denotation.

This is what I will mean by a 'representation' in the discussion to follow: an interpretation of (linguistic) sentences involving (M, g_{ab}) in terms of (physical) facts about the world that makes those sentences at least approximately true. And when I say that two linguistic structures can or cannot represent the same thing 'at once', I will mean this in the sense of being part of the same concrete interpretation. This is entirely standard practice: it is exactly what one finds in a typical general relativity textbook, and it is a standard way of thinking in both philosophy and physics. But it is worth being clear about this when we turn to characterising putative 'rules' for representation like Leibniz Equivalence.

2.2.2 Leibniz Equivalence in Spacetime Physics

When Earman and Norton originally defined 'Leibniz Equivalence' in the context of spacetime physics, they wrote:

> "*Leibniz Equivalence* Diffeomorphic models represent the same physical situation"

[6, p. 522]. Their phrase "diffeomorphic models" should be understood in our language to mean, "isometric Lorentzian manifolds". But their statement that two such models "represent the same physical situation" introduces the ambiguity we have discussed above.

Let me first head off a possible confusion: although a natural reading of Earman and Norton's words would be that isometric Lorentzian manifolds 'do' or 'must' represent the same physical situation, this would lead to a principle that is totally implausible. For example, two isometric Lorentzian manifolds (M, g_{ab}) and $(\tilde{M}, \tilde{g}_{ab})$

[5] This is pointed out by Roberts [22] and by Fletcher [8, § 3.2]

may both be isometric to the Schwarzschild solution but, as we saw above, these linguistic structures can still be used to represent different physical situations. This isn't particularly interesting, and it isn't what Earman and Norton had in mind.

What they more likely meant is that isometric models 'can' represent the same thing. But, this usage introduces an ambiguity, between the 'Weak' and 'Strong' readings of Leibniz Equivalence. That is, their statement could either mean:

- *Weak Leibniz Equivalence* Isometric Lorentzian manifolds can each be used with equal accuracy to represent a given physical situation (though not necessarily at once); or
- *Strong Leibniz Equivalence:* Isometric Lorentzian manifolds can all be used with equal accuracy to represent a given physical situation, at once.

Weak Leibniz Equivalence is irrelevant for the substantivalist debate: a substantivalist has no problem accepting that isometric descriptions can be used to describe the same physical situation in some contexts; they are mere alternative labellings for the same 'real' spacetime points.

In contrast, Strong Leibniz Equivalence is relevant to the substantivalism debate. Further confirmation of this reading can be found in the textbooks that Earman and Norton refer to in support of their definition, where the term "model" is used to indicate representation via a concrete interpretation: "the model for space-time is not just one pair (M, g) but a whole equivalence class of all pairs (M', g') which are [isometrically] equivalent to (M, g)" [11, p. 56]; and, "relativistic models are regarded as defined only up to isomorphism" [25]. More importantly, as we will see in the next section, only Strong Leibniz Equivalence allows for the Leibniz-inspired argument against substantivalism known as the 'hole argument'.

So, except where I specify otherwise, I will drop the 'Strong' prefix, and use the phrase 'Leibniz Equivalence' as shorthand for 'Strong Leibniz Equivalence':

Definition 3 (*Leibniz Equivalence*) Any pair of isometric Lorentzian manifolds (M, g_{ab}) and $(\tilde{M}, \tilde{g}_{ab})$ can be concretely interpreted to represent the same physical situation at once.

I will also sometimes have occasion to refer to its logical negation:

Definition 4 (*Leibniz Equivalence Denial*) A pair of isometric Lorentzian manifolds (M, g_{ab}) and $(\tilde{M}, \tilde{g}_{ab})$ exist that cannot be concretely interpreted to represent the same physical situation at once.

3 Leibniz Equivalence Fails

Only Leibniz Equivalence in the 'Strong' sense is relevant for the hole argument. But, in spite of the textbooks, and in spite of the common wisdom about diffeomorphism freedom in general relativity, Leibniz Equivalence (in the 'Strong' sense) is

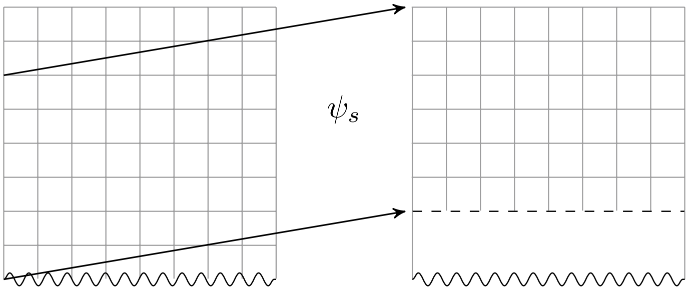

Fig. 1 Isometric embedding into a proper subset

false. There are situations in which Leibniz Equivalence can reasonably fail, independently of anything to do with substantivalism.

Here is a counterexample to show this. Consider the open half-plane $M = \mathbb{R} \times (0, +\infty)$ with a metric field g_{ab} that is the same at every point. The half-plane is infinite to the West, East and North, but finite to the South where there is an open boundary. This is a perfectly legitimate Lorentzian manifold, and even useful for modelling purposes. So, it can be concretely interpreted to represent some physical situation. If Strong Leibniz Equivalence were true, or if Hawking and Ellis [11, p. 56] were right, then all Lorentzian manifolds isometric to it could accurately represent the same physical situation at once.

But it isn't true.[6] This Lorentzian manifold is isometric to a proper subset of itself: translate the half-plane 'up' into itself by an arbitrary amount and the result is an isometry, as illustrated in Fig. 1. Call the image of that isometry $(\tilde{M}, \tilde{g}_{ab})$. It is a 'proper part' of the original half-plane, in the sense of being defined by a restriction to a proper subset (a 'Lorentzian submanifold'). So, it only represents a 'part' of the physical situation that the original half-plane represents. In most modelling scenarios, a 'whole' and its 'proper part' are not the same physical situation. So long as this is the case, the half-plane leads to the denial of Leibniz Equivalence: this pair of isometric Lorentzian manifolds cannot be concretely interpreted to represent the same physical situation at once.

The property leading to Leibniz Equivalence denial here can be formulated as a general principle, if one were in the business of proposing principles for good scientific representation (I am not):

Definition 5 (*The Principle of Composition*) A ('whole') Lorentzian manifold and one of its proper ('part') Lorentzian submanifolds cannot represent the same physical situation at once.

[6] Norton [18, §10.3.2] is well-aware of the failure of Leibniz Equivalence in general, although I think the example of this section makes it particularly perspicuous.

Rather than advocate any such general principle, I would like to point out that the half-plane is a perfectly legitimate model of general relativity, which can be used to represent physical situations for which the whole is different from any proper part. As a result, Leibniz Equivalence fails under reasonable circumstances. No assumption about spacetime substantivalism is needed to make this point: whether one chooses to interpret the manifold as a relationist or as a substantivalist, Leibniz Equivalence fails here.

Of course, it is possible to avoid this kind of counterexample with a little qualification. That is, it is still possible to say: the half-plane (M, g_{ab}) and its (isometric) Lorentzian submanifold $(\tilde{M}, \tilde{g}_{ab})$ can represent the same physical situation at once, insofar as we ignore that one is a proper subset of the other. I will discuss this kind of response in more detail in Sect. 5. But in general, it is up to the applied mathematician whether these subset relations are ignored. Thus, our lesson is really: the success or failure of Leibniz Equivalence depends entirely on how we happen to use language to represent the physical world. Nothing about the representation described above is unreasonable from the perspective of applied mathematics; the requirement of (Strong) Leibniz Equivalence, on the other hand, is questionable.

4 The Hole Argument Revisited

Einstein grappled with the hole argument in his preliminary writing on general relativity, but it was Earman and Norton [6] who identified its significance for the substantivalism debate. In this section I will first review their discussion of manifold substantivalism, and clarify its definition. I will then review the hole argument, and point out that its target is not so much Leibniz Equivalence denial, but a particular definition of manifold substantivalism.

4.1 Defining Manifold Substantivalism

Earman and Norton [6, p. 522] call the denial of Leibniz Equivalence the ‘acid test’ for substantivalists. But we have just seen that Leibniz Equivalence fails, independently of anything to do with substantivalism. How are we to interpret this result? A careful review of Earman and Norton will find that Leibniz Equivalence is not really the issue; the target is a more direct expression of manifold substantivalism.

Earman and Norton’s discussion of spacetime substantivalism is inspired by the Leibniz and Clarke correspondence. They say:

> [i]f everything in the world were reflected East to West (or better, translated 3 feet East), retaining all the relations between bodies, would we have a different world? The substantivalist must answer yes since all the bodies of the world are now in different spatial locations, even though the relations between them are unchanged. [6, p. 521].

For example: fix a concrete interpretation in which (M, g_{ab}) accurately represents a physical situation—say, spacetime around the Milky Way galaxy—and a “translation

3 feet East" is represented by a diffeomorphism $\psi : M \to M$, which drags along the metric to produce a new model, $(M, \psi^* g_{ab})$. This model is isometric to the first, "retaining all the relations between bodies" in the sense of test particles. But a substantivalist, according to Earman and Norton, will say that $(M, \psi^* g_{ab})$ and (M, g_{ab}) cannot both accurately represent the same physical situation at once. That is: substantivalists must deny Leibniz Equivalence.

Why can't the substantivalist take $(M, \psi^* g_{ab})$ and (M, g_{ab}) to represent the same thing at once? Because (Earman and Norton argue) they are committed to a "realist view" of the manifold M. In particular: to represent the same thing at once, both $(M, \psi^* g_{ab})$ and (M, g_{ab}) would be given a single concrete interpretation in which a point $p \in M$ represents a real physical event in spacetime. For example, let (M, g_{ab}) be the Schwarzschild solution interpreted to represent our galaxy, and let p be a point on the event horizon defined by g_{ab}. Then p will not generally be a point on the event horizon defined by $\psi^* g_{ab}$. So, the manifold substantivalist will deny that (M, g_{ab}) and $(M, \psi^* g_{ab})$ represent the same physical facts at once, in spite of the fact that they are isometric.

This is perfectly compatible with the claim that these two spacetimes have the same representational capacities, in the sense of Weak Leibniz Equivalence. The latter only implies that we could just as well have chosen $(M, \psi^* g_{ab})$ to represent our galaxy in the first place. But, once a concrete interpretation of (M, g_{ab}) is fixed, then a manifold substantivalist cannot at the same time use $(M, \psi^* g_{ab})$ to represent the same thing unless $\psi : M \to M$ is the identity transformation.

Earman and Norton thus implicitly assume that the manifold substantivalist is committed to:

Definition 6 (*manifold substantivalism*) Two Lorentzian manifolds (M, g_{ab}) and $(M, \tilde{g}_{ab})$ (with the same manifold M) can be concretely interpreted to represent some physical situation at once if and only if $g_{ab} = \tilde{g}_{ab}$.

Manifold substantivalism implies the denial of (Strong) Leibniz Equivalence. For, although a Lorentzian manifold (M, g_{ab}) and its 'shifted' counterpart $(M, \psi^* g_{ab})$ are isometric, the manifold substantivalist will deny that they represent the same thing at once, if (and only if) $\psi^* g_{ab} \neq g_{ab}$. However, it would be wrong to say that the most significant implication of the hole argument is to rebuke Leibniz Equivalence denial. As we will now see, it is not Leibniz Equivalence denial, but the definition of manifold substantivalism above that is the successful target of the hole argument.

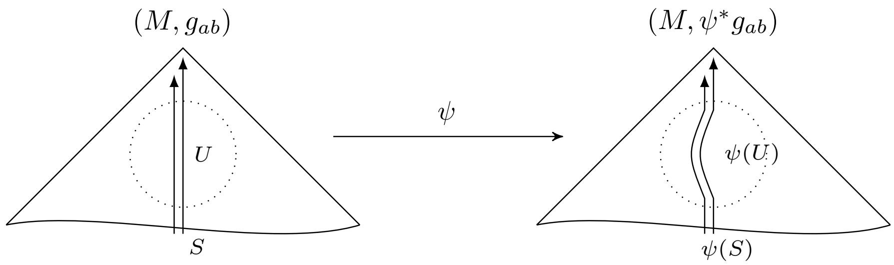

Fig. 2 The hole transformation

4.2 The Hole Argument

Let (M, g_{ab}) be a Lorentzian manifold with a Cauchy surface S and a time orientation[7]. Not all Lorentzian manifolds admit a Cauchy surface, but there always exists a subregion $N \subseteq M$ around each point such that $(N, g_{ab}|_N)$ is a Lorentzian manifold that does; so, just choose any one of them. Consider an open region U in the future domain of dependence of S, sometimes called the *hole region*. Let $\psi : M \to M$ be a diffeomorphism that is not the identity inside U, but which is the identity everywhere else, as depicted in Fig. 2. The two Lorentzian manifolds (M, g_{ab}) and $(M, \psi^* g_{ab})$ are then said to be related by a *hole transformation*. They are obviously isometric.

Suppose (M, g_{ab}) is concretely interpreted to represent some physical situation, such as the motion of two test particles. According to manifold substantivalism, it is only outside the region U that (M, g_{ab}) and $(M, \psi^* g_{ab})$ can be interpreted to represent the same physical situation at once—that is, in a single interpretation that identifies points in M with physical spacetime points—since that is the only region where ψ is the identity. The problem with this, Earman and Norton point out, is that it leads to "radical local indeterminism". For it means that the two distinct models (M, g_{ab}) and $(M, \psi^* g_{ab})$ can be interpreted as representing the same physical situation on S, but different situations in its future domain of dependence. The result is a classic expression of indeterminism: the present facts do not determine the future.[8] Earman and Norton interpreted this as a *reductio ad absurdum*: manifold substantivalism, as it is defined above, introduces radical local indeterminism into an otherwise perfectly good theory. This, they take it, is an absurdity worthy of rejection.

As it happens, manifold substantivalism also implies the denial of Leibniz Equivalence. Equivalently, accepting Leibniz Equivalence implies the rejection of manifold substantivalism; thus, Norton and Earman write, "[o]f course this radical local

[7] A *Cauchy surface* for a Lorentzian manifold (M, g_{ab}) is a spacelike hypersurface that is intersected by every inextendible timelike and null curve exactly once. A *time orientation* is a smooth timelike vector field, which serves to pick out one or the other light cone lobe as 'future-directed' at every point. Every Lorentzian manifold with a Cauchy surface can be given a time orientation.

[8] The result can be avoided by denying that (M, g_{ab}) and $(M, \psi^* g_{ab})$ ever represent the same points in physical spacetime, in spite of being isometric; this strategy was defended by Butterfield [4].

indeterminism can be escaped easily by just accepting Leibniz equivalence" [6, p. 524]. The intended implication is that, by adopting Leibniz Equivalence, one automatically rejects the problematic assumption of manifold substantivalism. Unfortunately, this suggestion is misleading: we may have independent reason to deny Leibniz Equivalence, as I have argued above. However, this does nothing to undermine the hole argument. And indeed, in their more careful moments, Earman and Norton rely on a more modest and appropriate claim: "the affected... substantivalists must deny Leibniz equivalence at least for hole diffeomorphisms, which already is sufficient to yield the dilemmas" [6, p. 522, fn.2]. The situation can be summarised:

$$\text{manifold substantivalism} \begin{array}{l} \Nearrow \ \text{radical indeterminism} \\ \\ \Searrow \ \text{Leibniz Equivalence Denial} \end{array}$$

This is the central point that I would like to make here: the hole argument says that rejecting radical indeterminism leads to the rejection of manifold substantivalism; but this says nothing about the status of Leibniz Equivalence. And, as I have argued above, Leibniz Equivalence as a general principle can be reasonably rejected on independent grounds.

5 Regarding Weatherall's Critique

Weatherall [27, p. 330] has recently suggested that the hole argument, at least as it is presented by Earman and Norton [6], "is based on a misleading use of the mathematical formalism of general relativity." His strategy is to argue that the mathematics of general relativity does not allow Leibniz Equivalence Denial to be coherently formulated: "the fact that such an isometry exists provides the only sense in which the two spacetimes are empirically equivalent" [27, p.337]. If true, this would make trouble not only for the hole argument, but for my claim that Leibniz Equivalence is in general false. In this section, I begin by reviewing Weatherall's general argument strategy in light of the account of Leibniz Equivalence developed above. I then show how the counterexamples to Leibniz Equivalence given above introduce a challenge for Weatherall as well.

5.1 Leibniz Equivalence as a Background Commitment

Weatherall's reasoning stems from what he calls a "background commitment", which he proposes to adopt when using mathematics to represent the physical world, and which he says leads the hole argument to be "blocked". This turns out to be none other than Leibniz Equivalence, in close to its original form:

> "isomorphic mathematical models in physics should be taken to have the same representational capacities. By this I mean that if a particular math-

ematical model may be used to represent a given physical situation, then any isomorphic model may be used to represent that situation equally well." [27, p. 332]

Weatherall applies this background commitment to the hole argument as follows: let (M, g_{ab}) and $(M, \psi^* g_{ab})$ be isometric Lorentzian manifolds related by a hole transformation. Then the background commitment implies that,

> "for any region R of spacetime that may be adequately represented by some region U of (M, g_{ab}), there is a corresponding region $\tilde{U} = \psi[U]$ of $(M, \tilde{g}_{ab})$ that can represent the same region of spacetime equally well, for all purposes" [27, p. 11].

Weatherall's expression of Leibniz Equivalence contains the ambiguity discussed in Sect. 2.1: it could be read as saying that isomorphic structures can represent the same thing at once (Strong Leibniz Equivalence), or not necessarily at once (Weak Leibniz Equivalence). The strength of Weatherall's intuitions about the commitment suggest that he has in mind 'Weak' Leibniz Equivalence, which is the more plausible principle. But Weak Leibniz Equivalence is irrelevant to the hole argument, as I have argued in Sect. 2.2. In contrast, the half-plane (among other examples) shows that 'Strong' Leibniz Equivalence is false, and therefore does not serve as a plausible background commitment about general relativity. That is, without further qualification, Weatherall's response to the hole argument faces a dilemma:

1. *Weak option:* adopt the potentially plausible principle of Weak Leibniz Equivalence, and make a statement that is irrelevant to the hole argument; or
2. *Strong option:* adopt the principle of Strong Leibniz Equivalence, which would render the hole argument unnecessary, but would be stating something false.

Weatherall and his defenders would seem to endorse an escape from this dilemma that proceeds by restricting the scope of Strong Leibniz Equivalennce, through a conditional statement analogous to Russell's escape from the paradoxes of naïve set theory. In the next subsection, I will try to clarify this response.

5.2 The Russellian Manoeuvre

Weatherall variously suggests that it would be ignominious to deny Leibniz Equivalence, since this would amount to a "misleading use of the formalism of general relativity" (p. 330), a failure to use this formalism "correctly, consistently, and according to our best understanding of the mathematics" (p. 330), and a view that is not "mathematically natural or philosophically satisfying" (p. 345). This approach might generate misgivings that Weatherall is just assuming by force of what he finds "philosophically satisfying" the very thing that Earman and Norton tried to establish by way of an argument. I would like to interpret these statements more charitably, as expressing a conditional assumption of Strong Leibniz Equivalence: *if a representation is adequate, then Strong Leibniz Equivalence must hold.* In short,

$$\text{Adequate representation} \;\Rightarrow\; \text{Strong Leibniz Equivalence}$$

or equivalently,

$$\text{Strong Leibniz Equivalence Denial} \;\Rightarrow\; \text{Inadequate representation.}$$

If one believes this conditional, then any representation that fails to satisfy Strong Leibniz Equivalence should simply be rejected as an inadequate representation. This again renders the hole argument at best irrelevant: manifold substantivalism implies the denial of Strong Leibniz Equivalence Denial; combining this with the above, one can conclude without further argument that such a representation is inadequate:

$$\text{Manifold substanvialism} \;\Rightarrow\; \text{Inadequate representation.}$$

This same strategy can be used to weasel out of counterexamples like the half-plane of Sect. 3. If a representation's adequacy requires Strong Leibniz Equivalence, then that example shows that a Lorentzian manifold is insufficient to represent the physical half-plane spacetime. For example, one might demand that the structure doing the representing is a Lorentzian manifold (M, g_{ab}) together with a preferred global coordinate system $\varphi : M \to \mathbb{R}^2$. Since this coordinate system is not preserved by the translation, in that $\varphi \circ \psi_s(x) \neq \varphi(x)$ for all $x \in M$, the isomorphism fails, and the counterexample to Strong Leibniz Equivalence is avoided.

Fletcher [8] has defended Weatherall through a rejoinder in this spirit. He begins by formulating an interpretive principle in the spirit of Weak Leibniz Equivalence, which he calls, Representational Equivalence by Mathematical Equivalence (REME): "If two models of a physical theory are mathematically equivalent, then they have the same representational capacities" [8, § 2]. Fletcher points out[9] that this principle by itself does little to avoid the indeterminism arising out of the hole argument, just as is argued at length in the discussion above (and in [22]). He follows this observation with his response, which is to restrict what kinds of properties a piece of mathematics can represent:

> The error implicit in this concern is the assumption that Lorentzian manifolds, as mathematical models, represent *all* properties of a physical relativistic spacetime (and its "contents"). ... Lorentzian manifolds may not exemplify properties of the states of affairs they represent, but all the properties they do exemplify—those not abstracted away—are the same for isomorphic [Lorentzian] manifolds. This is precisely encoded in the mathematical models themselves with the interpretation of isomorphic objects in a mathematical category as being equivalent as objects in that category. ... So any putative representational differences among isormorphic models, such as spacetime point haecceities, is

[9] Fletcher writes: "If two relativistic spacetimes, represented as Lorentzian manifolds, are related by a hole transformation, then they are isometric, i.e., related by a map that is an isomorphism in the category of Lorentzian manifolds. That means, by REME, that they have the same representational capacities, but not necessarily that they must represent the same unique physical state of affairs. Yet if this represented state of affairs is not unique, then the problem of indeterminism seems to rise again, forcing one to confront a metaphysical dilemma anew." [8, §5.2]

not reflected at all in the models themselves as members of category they are taken to be—there is no mathematical correlate of those differences definable in the category.

Fletcher is demanding that, if it is possible for a Lorentzian manifold to “exemplify” some properties—meaning, I take it, to adequately represent them—then those properties “are the same” for isometric Lorentzian manifolds. As a result, properties like “spacetime point haecceities”, sometimes taken to be associated with substantivalism, cannot be adequately represented using a Lorentzian manifold. I take this to amount to the same conditional statement of Strong Leibniz Equivalence formulated above: if a representation is adequate, then Strong Leibniz Equivalence must hold.

5.3 Trouble for the Russellian Manoeuvre

If one asserts that all adequate representations satisfy Leibniz Equivalence, then I agree that these conclusions follow. But, I have given a minimal account of adequate representations in Sect. 2.2.1 (following Curiel [5]) that does not require Leibniz Equivalence, and according to which the half-plane is a perfectly reasonable representation. In contrast, Weatherall does not suggest an argument for his background commitments, but rather states: “I will not to defend them here. Rather, I take them as background commitments that inform the arguments to follow” [27, p. 332]. Without such a defence, anyone (like me) who finds Lorentzian manifolds to adequately represent the physical half-plane will simply conclude that this new background commitment is also false. That is, if an adequate representation fails to satisfy Strong Leibniz Equivalence, then the manoeuvre fails:

$$\neg(\text{Adequate representation} \;\Rightarrow\; \text{Strong Leibniz Equivalence}).$$

Thus, the Russellian manoeuvre is subject to a challenge as well: rebuke the half-plane counterexample on independent grounds, without simply assuming by fiat that all adequate representations satisfy Strong Leibniz Equivalence. Or, more directly: provide independent reason to think that all adequate representations must satisfy Strong Leibniz Equivalence.

The challenge is made difficult by the fact that in the practice of science, representations commonly fail to satisfy Strong Leibniz Equivalence. For example, Einstein and Grossman’s original expression of the hole argument in 1913–1914 was actually an argument against Strong Leibniz Equivalence [17, §3]. Another counterexample is the half-plane Lorentzian manifold given above. Further counterexamples abound: Any two single-element sets are isomorphic to each other; but, one can use these sets to represent a black raven, or to represent a white shoe. So, Weak Leibniz Equivalence is satisfied, and the sets have the same ‘representational capacities’. But, one cannot use them both to represent the same thing at once, on pain of paradoxes of multiple denotation. And it is this second kind of comparison that is at issue in the statement of manifold substantivalism.

In general, Strong Leibniz Equivalence can fail in relativity theory whenever Lorentzian manifolds represent the nature of spacetime incompletely. Such examples are well-known [8, 22]. For example, one cannot always neglect the interaction

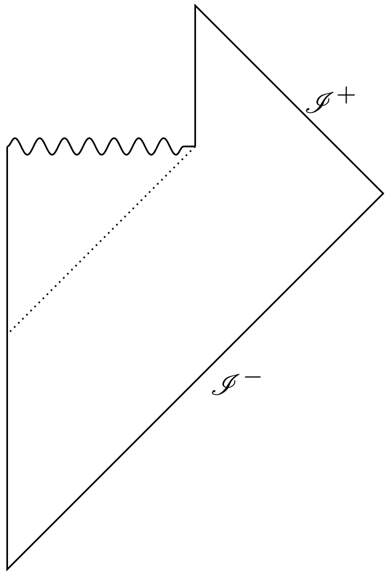

Fig. 3 The Lorentzian manifold associated with an evaporating black hole has many inequivalent interpretations, associated with the many current proposals for black hole microphysics

between spacetime and quantum fields, as in the analysis of black hole thermodynamics. Nevertheless, we often use Lorentzian manifolds to represent such situations, as Hawking [12, p. 219] did in his famous depiction of an evaporating black hole, illustrated here in Figure 3. There is currently still considerable disagreement about what such spacetimes represent, for example about whether quantum fields evolve unitarily between hypersurfaces [2, 14]. That is, the same Lorentzian manifold (M, g_{ab}) can of course represent many different situations! But this should not dissuade anyone from the practice of representing spacetime using a Lorentzian manifold. We must simply admit that two isometric spacetimes—in this case, the very same spacetime—may sometimes be used to represent different physical situations, depending on what we choose to represent.

A rejoinder could be to argue that in such cases, the category of Lorentzian manifolds is not the "adequate" mathematical device for representing spacetime: different approaches to quantum gravity will adopt different categories to capture the complete facts about spacetime in a more adequate way, which breaks the isomorphism. This response is difficult to square with the fact that, as a matter of practice, we do use Lorentzian manifolds to represent such things, and we generally accept that our physical descriptions are incomplete. Wigner thought that "laws of nature contain, in even their remotest consequences, only a small part of our knowledge of the inanimate world" [29, p. 5]. This kind of incompleteness does not prevent us from legitimately representing spacetime using a Lorentzian manifold. More importantly, such a response just pushes the problem to one of determining the appropriate category for the practice of representing spacetime.

The answer to this problem is not automatic, nor can it be derived mathematically. How mathematics is used to represent the physical world is defined by human experience and human intention. It is this freedom that is responsible for the fact that isometry is not "the only way" for us to determine whether two Lorentzian manifolds can represent the same physical situation at once. Einstein's use of mathematics in the early formulation of general relativity violated Leibniz Equivalence. There is no *a priori* reason to prohibit such thinking, or to prohibit the more general practice of free choice in defining when two mathematical structures represent the same situation.

5.4 The Identity Map Argument

Part of Weatherall's discussion is an observation about the differing equivalence relations in play in the hole argument. For completeness, let me now offer a few comments on that discussion.

One of the consequences of manifold substantivalism is that, if $\psi : M \to M$ is a diffeomorphism, then (M, g_{ab}) and $(M, \psi^* g_{ab})$ can represent the same physical situation at once if and only if $\psi^* g_{ab} = g_{ab}$, i.e. when ψ is an isometry of (M, g_{ab}). But the Lorentzian manifolds used in realistic modelling situations do not have non-trivial isometries, which means that ψ can only be the identity map: $\psi(p) = p$ for all $p \in M$. Weatherall refers to this map as 1_M, to emphasise that it is the identity element in the category of manifolds. So, manifold substantivalism is at least committed to the view that, in realistic modelling situations, the two spacetimes (M, g_{ab}) and $(M, \psi^* g_{ab})$ can represent the same physical situation at once if and only if they are related by the manifold identity map 1_M. We might even wish to call this a commitment about 'physical equivalence' using 1_M.

Weatherall argues that this standard of physical equivalence is confused. To warm us up to this conclusion, he gives the following 'warmup exercise': consider two groups constructed from the set of integers $\mathbb{Z}$, denoted by $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \tilde{+})$. The binary operation '$+$' of the first group is normal arithmetic addition, so that $3 + 5 = 8$, and so on. But the binary operation '$\tilde{+}$' of the second group is arithmetic addition followed by subtraction of 1, so that $3 \tilde{+} 5 = 7$, and in general $n \tilde{+} m = n + m - 1$. The identity element in the first group is 0, while the identity element in the second is 1. Suppose someone now cries out, 'These groups have different identity elements!' What are we to make of this poor, confused exclamation?

Weatherall replies that, since the two groups are obviously isomorphic, "there is no ambiguity regarding the additive identity of the integers", since after all, "the identity element is provably unique for any group" [27, p. 333]. In asserting that 0 and 1 are different identity elements, the confused exclamation assumed that the identity map $1_{\mathbb{Z}}$ (defined by $1_{\mathbb{Z}}(n) = n$ for all $n \in \mathbb{Z}$) is a reasonable way to compare the two groups. But $1_{\mathbb{Z}}$ is not an isomorphism between $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \tilde{+})$, and so "the identity map is not the relevant standard of comparison" [27, p. 334].

The general principle that Weatherall proposes is thus: we should be sure to adopt an adequate equivalence relation "given by the mathematics used in formulating those models" [27, p. 331][10]. Earman and Norton's manifold substantivalist is

[10] And again, in his discussion of Newtonian spacetimes: "the relevant standard of equivalence is already manifest in the map that relates the structures in the first place" [27, p. 342].

making an illegitimate step in using the manifold identity 1_M to judge the equivalence of (M, g_{ab}) and $(M, \psi^* g_{ab})$, according to this view. These Lorentzian manifolds are isomorphic, in the sense of being related by an isometry, even though the manifold identity 1_M does not describe an isometry between them. Like the confused exclamation about the group identity element, the manifold substantivalist exclaims that "there is an ambiguity with regard to the value of the metric at a point $p \in O$ at which the two metrics disagree" [27, p. 337]. The problem, Weatherall says, is: "the fact that such an isometry exists provides the *only* sense in which the two spacetimes are empirically equivalent" [27, p. 337]—and thus, "the two Lorentzian manifolds agree on the metric at every point—there is no ambiguity, and no indeterminism" [27, p. 338].

In a purely mathematical context, I am happy enough to agree to the demand that we use group-isomorphism to compare groups, or isometry to compare Lorentzian manifolds.[11] But that is an agreement about totally linguistic structures, which says nothing about when such structures can be used to accurately represent the same physical situation at once. A further assumption is needed to restrict what a mathematical language represents, such as Weak or Strong Leibniz Equivalence. And, as I have argued above, neither principle succeeds in refuting the hole argument, even with the Russellian manoeuvre of restricting adequate representations.

This same critique applies to both groups of integers and Lorentzian manifolds. For example, suppose we use the integers to label an infinite array of classical particles set out in physical space. Then it is not at all clear that the groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \tilde{+})$ represent can represent the same thing at once, exactly because (for example) they disagree about which label represents the additive identity. The same response is available to the defender of the hole argument: isometry alone says nothing about whether or not two Lorentzian manifolds can be used to represent the same physical situation at once. As a result, the manifold substantivalist's distinction between them is illegitimate not because of a principle of applied mathematics, but because of the hole argument.

5.5 An Alternative Brand of Quietism

There is a gentler brand of quietism in the neighbourhood of Weatherall's view that might be worth clarifying. It is an attitude that I myself adopt from time to time, and which provides some guidance on how to react to the hole argument. The main difference is that this view will be presented as a mere attitude, as opposed to a rule restricting the use of mathematical representations. I know of no argument that establishes the present view. Some simply take comfort in the gentle, Buddhist-like perspective on the philosophy of physics that this attitude provides.

[11] That said, the mathematical platonist may disagree, for reasons similar to my concerns here. For the platonist, a number like (say) zero may have real existence independently of the human mind, and it is this object that we represent using the linguistic symbol 0. For such a platonist, the two groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \tilde{+})$ cannot describe mathematical reality with equal accuracy at once.

The attitude begins by stating the propositions that we have good evidence to believe, in the normal language of science[12]. For example, we may all agree that the region near the galactic centre has the structure of Kerr spacetime. But at this point, the attitude refuses all further interpretive claims. Questions like ‘Is the set of spacetime points real?’ are passed over silently. In their stead one adopts an attitude of quietism as far as the propositions of realism about unobservables are concerned.

I take this to capture a sense of what Arthur Fine has called the Natural Ontological Attitude (NOA), which he summarises as the recommendation to “try to take science on its own terms, and try not to read things into science” [7, p. 149]. This perspective can be helpful, and indeed I often find myself joining its practitioners in the monastery for a little peace of mind. However, there is no use pretending that this view is established by any rigorous argument or rule, as Fine is quick to point out:

> It does not comprise a doctrine, nor does it set a philosophical agenda. At most it orients us somewhat on how to pursue problems of interest, promoting some issues relative to others just because they more clearly connect with science itself. Such a redirection is exactly what we want and expect from an attitude, which is all that NOA advertises itself as being. [7, p. 10].

The NOA attitude toward manifold substantivalism, I take it, is often an exercise in the discipline of silence.

However, the hole argument is not necessarily a case where this attitude is appropriate. The argument itself identifies an interesting connection between the realism debate and philosophy of science, in establishing a link between manifold substantivalism and indeterminism. It has also promoted interesting connections between the realism debate and modern physics in helping to motivate a relationist perspective on spacetime in quantum gravity [1, 15, 23] as well as sophisticated substantivalist alternatives [e.g. 19]. These commentators did not simply fail to “recognize the mathematical significance of an isomorphism”[27, p. 339 fn. 22]. On the contrary, with too much quietism you may miss out on some of the fun.

6 Conclusion

Leibniz Equivalence is not as clear as it might first seem. Its plausibility depends on whether we take two isomorphic mathematical structures to be capable of representing the same physical situation ‘at once’. If not necessarily (Weak Leibniz Equivalence), then the principle expresses something plausible about the representational capacities of those structures, but which is irrelevant to the hole argument. If so (Strong Leibniz Equivalence), then the principle is false by explicit counterexample. Fortunately, this by itself does not threaten the hole argument, which targets a more specific statement of manifold substantivalism. But, it does threaten the argument of Weatherall and others who have defended a version of Leibniz Equivalence. I have argued Weak Leibniz Equivalence is irrelevant to the hole argument, and

[12] This is akin to what Ruetsche [24, §1] refers to as a *partial* interpretation of a theory.

Strong Leibniz Equivalence is false. And, although one might try to save the latter by demanding that it only fails in inadequate representations, I have argued that this goes against the ordinary practice of science.

Thus, in spite of some recent threats, Earman and Norton's hole argument remains: one pays a high price for a belief in manifold substantivalism. To that, I would now add: the price may be no less for unwarranted principles of applied mathematics.

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References

1. Belot, G., Earman, J.: From metaphysics to physics. In: Butterfield, J., Pagonis, C. (eds.) From Physics to Philosophy. Cambridge University Press, Cambridge (1999)
2. Belot, G., Earman, J., Ruetsche, L.: The Hawking information loss paradox: the anatomy of controversy. Br. J. Philos. Sci. **50**(2), 189–229 (1999)
3. Brighouse, C.: Spacetime and holes. In: PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association, vol. 1994, pp. 117–125. Philosophy of Science Association (1994)
4. Butterfield, J.: The hole truth. Br. J. Philos. Sci. **40**(1), 1–28 (1989)
5. Curiel, Erik: General relativity needs no interpretation. Philos. Sci. **76**(1), 44–72 (2009). http://philsci-archive.pitt.edu/4567/
6. Earman, J., Norton, J.D.: What price spacetime substantivalism? The hole story. Br. J. Philos. Sci. **38**, 515–525 (1987)
7. Fine, A.: The Shaky Game: Einstein, Realism and the Quantum Theory. The University of Chicago Press, Chicago, London (1986)
8. Fletcher, S.C.: On representational capacities, with an application to general relativity. Foundations of Physics, pp. 1–22 (2018)
9. Goodman, N.: Languages of Art: An Approach to a Theory of Symbols. The Bobbs-Merrill Company Inc, Indianapolis, New York, Kansas City (1968)
10. Gryb, S. Thébault, K.P.Y.: Regarding the 'hole argument'and the 'problem of time'. Philos. Sci. **83**(4), 563–584 (2016). Preprint: http://philsci-archive.pitt.edu/11795/
11. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-Time. Cambridge University Press, New York (1973)
12. Hawking, S.W.: Particle creation by black holes. Commun. Math. Phys. **43**(3), 199–220 (1975)
13. Hodges, W.: Model theory. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy, fall 2018 edn. Metaphysics Research Lab, Stanford University (2018). https://plato.stanford.edu/archives/fall2018/entries/model-theory/
14. Hossenfelder, S., Smolin, L.: Conservative solutions to the black hole information problem. Phys. Rev. D **81**(6), 064009 (2010). Preprint: https://arxiv.org/abs/0901.3156

15. Isham, C.J.: Canonical quantum gravity and the problem of time. In: Integrable systems, quantum groups, and quantum field theories, pp. 157–287. Springer (1993). arXiv:gr-qc/9210011
16. Leibniz, G.W., Clarke, S.: Correspondence. Hackett Publishing Company, Inc, Indianapolis, IN (2000). Open Access Version: https://archive.org/details/leibnizclarkecor00clar
17. Norton, J.D.: Einstein, the hole argument and the reality of space. In: Forge, J. (ed.) Measurement, Realism and Objectivity: Essays on Measurement in the Social and Physical Sciences, pp. 153–188. D. Reidel Publishing Company, Dordrecht, Holland (1987)
18. Norton, J.D.: The hole argument. In: Zalta, Edward N. (ed.) The Stanford Encyclopedia of Philosophy, Summer 2019 edn. Metaphysics Research Lab, Stanford University (2019). https://plato.stanford.edu/entries/spacetime-holearg/
19. Pooley, O.: Points, particles, and structural realism. In: Rickles, D., French, S., Saatsi, J. (eds.) The Structural Foundations of Quantum Gravity, pp. 83–120. Oxford University Press, New York (2006a). Preprint: http://philsci-archive.pitt.edu/2939/
20. Pooley, O.: A hole revolution, or are we back where we started? Stud. Hist. Philos. Mod. Phys. **37**(2), 372–380 (2006b)
21. Priest, G.: Multiple denotation, ambiguity, and the strange case of the missing amoeba. Logique Anal **38**(150/152), 361–373 (1995)
22. Roberts, B.W.: Disregarding the hole argument (2014). Unpublished Preprint: http://philsci-archive.pitt.edu/11687/
23. Rovelli, C.: Quantum Gravity. Cambridge monographs on mathematical physics. Cambridge University Press, Cambridge (2004)
24. Ruetsche, L.: Interpreting Quantum Theories. Oxford University Press, New York (2011)
25. Sachs, R.K., Wu, H.-H.: General Relativity for Mathematicians. New York Inc., Springer Verlag (1977)
26. Saunders, S.: Physics and Leibniz's principles. In: Brading, K., Castellani, E. (eds.) Symmetries in Physics: Philosophical Reflections, chapter 16, pp. 289–307. Cambridge University Press, Cambridge (2003). Preprint: http://philsci-archive.pitt.edu/2012/
27. Weatherall, J.O.: Regarding the 'hole argument'. Br. J. Philos. Sci. **69**(2), 329–350 (2018). Preprint: http://philsci-archive.pitt.edu/11578/
28. Weisberg, M.: Simulation and similarity: Using models to understand the world. Oxford University Press, New York City (2013)
29. Wigner, E.P.: The unreasonable effectiveness of mathematics in the natural sciences. Commun. Pure Appl. Math. **13**(1), 1–14 (1960). Richard Courant lecture in mathematical sciences delivered at New York University, May 11, 1959
30. Wilson, M.: Wandering Significance: An Essay on Conceptual Behavior. Oxford University Press, New York (2006)