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EXPLICIT ASYMPTOTICS ON FIRST PASSAGE TIMES OF DIFFUSION PROCESSES

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Abstract

We introduce a unified framework for solving first passage times of time-homogeneous diffusion processes. According to the potential theory and the perturbation theory, we are able to deduce closed-form truncated probability densities, as asymptotics or approximations to the original first passage time densities, for the single-side level crossing problems. The framework is applicable to diffusion processes with continuous drift functions; especially, for bounded drift functions, we show that the perturbation series converges. In the present paper, we demonstrate examples of applying our framework to the Ornstein-Uhlenbeck, Bessel, exponential-Shiryaev (studied in [13]), and the hypergeometric diffusion [8] processes. The purpose of this paper is to provide a fast and accurate approach to estimate first passage time densities of various diffusion processes.

Keywords: First Passage Time; Diffusion Process; Perturbation Theory; Ornstein-Uhlenbeck Process; Bessel Process; Exponential-Shiryaev Process; Hypergeometric Diffusion; Special Functions

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1. Introduction

The interest of understanding the *first passage time* (FPT) could be traced back to the early 20th century [4, 46]. Known also as the *first hitting time*, the FPT defines a random time that a stochastic process would visit a predefined state. The phenomenon of uncertainty in time is often observed from natural or social science. Therefore, within a century the FPT has been actively studied in economics, physics, biology, etc. [14, 37, 43, 45].

Depending on various types of underlying processes and hitting boundaries, the FPT itself consists of a large cluster of different research. We refer to [3, 7, 38, 51] for a non-conclusive review. Among those research, especially in the area of mathematical finance and insurance, single-side constant-barrier crossing problem is one of the most commonly studied, e.g. [5, 17]. A general approach for solving such problem starts with finding the *Laplace transform* (LT) of the FPT density (FPTD). The LT usually comes from a unique solution to a second order non-homogeneous ODE with Dirichlet-type boundary values [18, 26]. For many familiar diffusion processes, the LTs have been solved and are listed in [9]. However, those LTs usually are expressed in terms of special functions and only a few of them have explicit inverse transforms. Therefore, many efforts have been made on the numerical inverse side. We refer to [1] for more details. Alternatively, using spectral theorem on linear operators [27, 30, 31], one can simplify the original LT. Under certain circumstances, closed-form FPTDs could be acquired through series representations [2, 34]. But people may find that the spectral decomposition approach has convergence issue for small t. In the present paper, our object is to apply the perturbation theory and solve explicit FPTD approximations for general single-side level crossing problems.

Consider a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ which satisfies the usual conditions and is generated by a standard Brownian motion. Let I be an open interval on $\mathbb{R}$ and $h(\cdot)$ be a real-valued continuous function defined on I. Our underlying process is from a class of SDEs which have at least weak solutions and are strong Markov:

$$X_t = \epsilon h(X_t)dt + dW_t, \;\; X_0 = x \in I. \tag{1}$$

Under our settings, $\epsilon \in \mathbb{R}$ and it should properly define $\{X_t\}_{t\geq 0}$ on its domain. For the convenience of deduction, we set the volatility to be constant. If a time-homogeneous diffusion coefficient $\sigma(x)$ is given, one may refer to [44, Theorem 1.6] to retrieve an SDE in (1) by using a time-changed Brownian motion. Also, consider a hitting level $a \in \mathbb{R}$, we specify two types of boundaries on I:

$$\partial I_a^u := \{a, +\infty\}\,,\ \partial I_a^l := \{-\infty, a\}\,,$$

namely boundaries for upper- and lower-regions. For shorthand, we use ∂I_a to represent single-side boundaries without labelling the direction. By suppressing x and a, we define the FPT of $\{X_t\}_{t\geq 0}$ from x to a through

$$\tau := \inf\{t > 0 : X_t \in \partial I_a\}\,.$$

Note that the Brownian filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is continuous. Therefore according to [42], τ is well defined (i.e. *regular* at ∂I_a). In addition, for $x \in I$, it is guaranteed that $\mathbb{P}_x(\tau > 0) = 1$.

For those FPTs which are *almost surely* (*a.s.*) finite, *i.e.* $\mathbb{P}_x(\tau < +\infty) = 1$, we are interested in acquiring their explicit distributions. Clearly, when $h(x) \equiv 0$ (a standard Brownian motion) the distribution of τ is given by inverse Gaussian (or inverse Gamma, equivalently) [9]. However, for most of non-trivial drifts, there is no closed-form solution. An example is $h(x) = x$ and which corresponds to the Ornstein-Uhlenbeck (OU) process. In this case, the explicit density is only available by restricting $a = 0$ [20].

In this paper, we apply the perturbation technique [24] to solve Dirichlet-type *boundary value problems* (BVPs). By inverting the perturbed LTs from the frequency domain, where those LTs usually have much simpler forms, we then are able to derive closed-form asymptotic densities in the time domain. The main contribution of this paper is to provide a unified recursive framework for solving the single-side barrier hitting problem. By using the Green's function representation and the potential theory [42], we prove the convergence of the perturbation series and demonstrate the error of a truncated series, respectively. As illustrations, we show the perturbed FPTDs of the OU, the Bessel, the exponential-Shiryaev [13], and the hypergeometric diffusion processes [8] in this paper, alongside their numerical comparisons with inverse Laplace transform algorithms [1] and Monte Carlo simulations.

The rest of the paper is organised as follows: Section 2 introduces our main results;

Section 3 demonstrates applications on various diffusion processes; in Section 4 we provide numerical comparisons; Section 5 concludes. Apart from where it is mentioned in the main body, all proofs can be found in [33] or the appendix of this paper.

2. Main Results

2.1. Perturbed Dirichlet Problem

We follow our previous settings. Further denote by $C^2 := C^2(I)$. For any $f \in C^2$, we also assume that the infinitesimal generator $\mathcal{A}f(x)$ of $\{X_t\}_{t\geq 0}$ exists for all $x \in I$:

$$\mathcal{A}f(x) = \epsilon h(x)f^{'}(x) + \mathcal{G}f(x),$$

where $\mathcal{G}\cdot$ is the infinitesimal generator for a standard Brownian motion:

$$\mathcal{G}f(x) = \frac{1}{2}f^{''}(x). \tag{2}$$

Consider $\beta \in \mathbb{C}^+$ (i.e. $\beta \in \mathbb{C}$ with $Real(\beta) > 0$), define,

$$f(x,\beta) := \mathbb{E}_x\left[e^{-\beta\tau}V\left(X_\tau\right)\right], \tag{3}$$

where $V(\cdot)$ is a finite function. The first step of our work is to find a proper BVP which is satisfied by $f(x,\beta)$. To see this, note that $\{X_t\}_{t\geq 0}$ is continuous over all stopping times; on the other hand, by our assumption $\{X_t\}_{t\geq 0}$ is a strong Markov process. According to the potential theory [42], $f(x,\beta)$ is the unique solution to the following Dirichlet problem:

$$\mathcal{A}f(x) = \beta f(x),\ x \in I. \tag{4}$$

Moreover, the corresponding boundary conditions are given by

$$f(\partial I_a) = \underline{V}. \tag{5}$$

In the notation above, $\underline{V} := [V(a), V(\pm\infty)]^T$ is a vector of the boundary values depending on the direction of crossing. Refer to (3), by setting $V(a) = 1$ and $V(\pm\infty) = 0$, we immediately find that the solution to the BVP (4) and (5) is the LT for the density of τ:

$$f(x,\beta) = \mathbb{E}_x\left[e^{-\beta\tau}\right].$$

In the second step, we apply perturbations on ϵ and find perturbed BVPs accordingly. The perturbation approach is a common technique in solving asymptotics for complex systems. It has been successfully applied in quantum physics and mathematical finance [16, 19, 47]. Traditionally, it is required that the perturbation parameter should be small. However, we will show later this is not necessary in our case.

For abbreviation, we ignore the function arguments in following contents. By default all operations are *w.r.t.* x. Consider a sequence of C^2-functions $\{f_i\}_{i\geq 0}$ such that f can be expressed as

$$f = \sum_{i=0}^{\infty} \epsilon^i f_i. \tag{6}$$

Substituting (6) into (4) yields

$$\sum_{i=0}^{\infty} \epsilon^i \left(\epsilon h f_i^{'} + \mathcal{G} f_i \right) = \sum_{i=0}^{\infty} \epsilon^i \beta f_i, \ \forall x \in I. \tag{7}$$

Rearranging terms in (7), we further get

$$\mathcal{G} f_0 - \beta f_0 + \sum_{i=1}^{\infty} \epsilon^i \left(h f_{i-1}^{'} + \mathcal{G} f_i - \beta f_i \right) = 0, \ \forall x \in I.$$

Note that, by extracting the 0-th order and assigning proper boundary conditions we can have the BVP for the standard Brownian motion (where the LT inverse is already known). Higher order BVPs can be solved via a recursive system which accumulates information from f_0 and the drift function h.

Denote the BVP with $i = 0$ by $o(1)$ term, by assigning the same boundary conditions as in the initial problem, we have

$$o(1) : \mathcal{G} f_0 = \beta f_0, \ x \in I$$
$$f_0(\partial I_a) = [1, 0]^T.$$

For $i \geq 1$, we use the notation $o(\epsilon^i)$ and define

$$o\left(\epsilon^i\right) : \mathcal{G} f_i = \beta f_i - h \cdot f_{i-1}^{'}, \ x \in I$$
$$f_i(\partial I_a) = [0, 0]^T.$$

In practice, it is not realistic of having infinite order solutions. We are more interested in knowing if the series (6) converges; or, given a truncation order $N \in \mathbb{N}$, what exactly the error terms are in the remained higher order ODEs. These two questions are answered, respectively, by the following two subsections.

2.2. Convergence of the Perturbation Series

Proposition 2.1. (Sufficient Conditions for the Convergence of the Perturbation.) *Let $\beta \in \mathbb{C}^+$ and h be the real-valued continuous drift function defined on I. If both h and h' are bounded on I and β is large enough, then the perturbation series* (6) *is convergent.*

Proof. *W.l.o.g.*, we consider the domain $I = (0, +\infty)$ (this requirement is only for the simplicity of presenting; depending on the hitting direction the domain can be chosen as $(a, +\infty)$ or $(-\infty, a)$).

Define G_β to be the *Green's operator* of $\beta - \mathcal{G}$ (cf. Eq. (2)) in I such that

$$G_\beta u(x) := \int_0^{+\infty} G_\beta(x,y)u(y)dy,$$

where $G_\beta(x,y)$ is the Green's function of the linear operator $\beta - \mathcal{G}$. By the definition of the Green's function, we can show that $G_\beta(x,y)$ is of the following explicit form:

$$G_\beta(x,y) = \begin{cases} \sqrt{\frac{2}{\beta}}\sinh(\sqrt{2\beta}y)e^{-\sqrt{2\beta}x}, \ y \leq x, \\ \sqrt{\frac{2}{\beta}}\sinh(\sqrt{2\beta}x)e^{-\sqrt{2\beta}y}, \ y > x. \end{cases}$$

We further introduce the multiplication operator $Hu(x) := h(x)u(x)$ and the derivative operator $Du(x) := u'(x)$. By considering the solution of $o(\epsilon^i)$-ODE and using the Green's function, one can check that

$$f_i = (G_\beta HD)^i f_0. \tag{8}$$

Substituting (8) into (6) further yields

$$f = \sum_{i=0}^{\infty}(\epsilon G_\beta HD)^i f_0. \tag{9}$$

As a sufficient condition for (9) converges, we only need to show that the norm of the operator $\epsilon G_\beta HD$ defined on a proper function space is less than one. To see this, consider the Banach space $L^\infty(I)$ and $u \in L^\infty(I)$. By definition,

$$||\epsilon G_\beta HDu||_\infty = ||\epsilon\sqrt{\frac{2}{\beta}}\Big(\int_0^x \sinh(\sqrt{2\beta}y)e^{-\sqrt{2\beta}x}h(y)u'(y)dy \tag{10}$$

$$+ \int_x^\infty \sinh(\sqrt{2\beta}x)e^{-\sqrt{2\beta}y}h(y)u'(y)dy\Big)||_\infty. \tag{11}$$

Using integration by parts, the integral in the right-hand side of (10) becomes

$$\begin{aligned}
&\int_0^x \sinh(\sqrt{2\beta}y)e^{-\sqrt{2\beta}x}h(y)u^{'}(y)dy \\
=&e^{-\sqrt{2\beta}x}\int_0^x (u(x)-u(z))\sqrt{2\beta}\cosh(\sqrt{2\beta}z)h(z)dz \qquad (12)\\
&+e^{-\sqrt{2\beta}x}\int_0^x (u(x)-u(z))\sinh(\sqrt{2\beta}z)h^{'}(z)dz. \qquad (13)
\end{aligned}$$

Similarly, the integral in (11) becomes

$$\begin{aligned}
&\int_x^\infty \sinh(\sqrt{2\beta}x)e^{-\sqrt{2\beta}y}h(y)u^{'}(y)dy \\
=&\sinh(\sqrt{2\beta}x)\int_x^\infty (u(z)-u(x))\sqrt{2\beta}e^{-\sqrt{2\beta}z}h(z)dz \qquad (14)\\
&-\sinh(\sqrt{2\beta}x)\int_x^\infty (u(z)-u(x))e^{-\sqrt{2\beta}y}h^{'}(y)dy. \qquad (15)
\end{aligned}$$

We observe that (12) is bounded by

$$\frac{1-e^{-2\sqrt{2\beta}x}}{2}(||u||_\infty - u(x))||h||_\infty, \qquad (16)$$

(13) is bounded by

$$\frac{1+e^{-2\sqrt{2\beta}x}-2e^{-\sqrt{2\beta}x}}{2\sqrt{2\beta}}(||u||_\infty - u(x))||h^{'}||_\infty, \qquad (17)$$

(14) is bounded by

$$\frac{1-e^{-2\sqrt{2\beta}x}}{2}(||u||_\infty - u(x))||h||_\infty, \qquad (18)$$

and (15) is bounded by

$$\frac{1-e^{-2\sqrt{2\beta}x}}{2\sqrt{2\beta}}(||u||_\infty - u(x))||h^{'}||_\infty. \qquad (19)$$

Now (16) and (18) together are bounded by

$$\left(1-e^{-2\sqrt{2\beta}x}\right)(||u||_\infty - u(x))||h||_\infty \le ||u||_\infty||h||_\infty$$

and (17) and (19) together are bounded by

$$\frac{2-2e^{-\sqrt{2\beta}x}}{2\sqrt{2\beta}}(||u||_\infty - u(x))||h^{'}||_\infty \le \frac{1}{\sqrt{2\beta}}||u||_\infty||h^{'}||_\infty.$$

Hence,

$$||\epsilon G_\beta HDu||_\infty \le \epsilon\left(\sqrt{\frac{2}{\beta}}||h||_\infty + \frac{1}{\beta}||h^{'}||_\infty\right)||u||_\infty,$$

and as long as h and $h^{'}$ are bounded on I and β is large enough, the norm of the operator is guaranteed to be less than one. This concludes our proof.

□

Remark 2.1. *Using the operator representation, Eq.* (8) *produces the solution to the* $o(\epsilon^i)$*-ODE. Alternatively, the solution can also be written recursively as*

$$f_i = f_0 \left[\int_0^x 2e^{2\sqrt{2\beta}y} \left(\int_0^y h(z)k_i(z,\beta)e^{-2\sqrt{2\beta}z} dz + C_1 \right) dy + C_2 \right],$$

where

$$k_i(z,\beta) := \sqrt{2\beta} f_0^{-1}(z,\beta) f_{i-1}(z,\beta) - \partial_z \left(f_0^{-1}(z,\beta) f_{i-1}(z,\beta) \right).$$

C_1 *and* C_2 *are constants subject to* $f_i(\partial I, \beta) = [0,0]^T$*. We refer the reader to [33, Lemma 3.4.2, p. 38] for more details of the recursive representation.*

2.3. Error Function

In this subsection, we provide an error function to the perturbed density function with a truncation order $N \in \mathbb{N}$.

Denote the N-th order truncated LT by

$$f^{(N)} := \sum_{i=0}^{N} \epsilon^i f_i.$$

We further assume that the inverse LTs of f, $f^{(N)}$, and $\partial_x f_N(x,\beta)$ exist, and which are denoted by $p_\tau(t)$, $p_\tau^{(N)}(t)$, and $\eta_N(x,t)$, respectively. Define the difference (absolute error) between the original and the perturbed FPTDs by

$$q_\tau(t) := p_\tau(t) - p_\tau^{(N)}(t). \tag{20}$$

Then we have the following result.

Proposition 2.2. (Probabilistic Representation of the Truncation Error.) *For all* $t \in (0,+\infty)$ *and all* $\beta \in \mathbb{C}^+$*, if*

$$\int_0^{+\infty} e^{-\beta t} \mathbb{E}_x \left[\int_0^{\tau \wedge t} |h(X_u) \eta_N(X_u, t-u)| \, du \right] dt < +\infty, \tag{21}$$

then

$$q_\tau(t) = \epsilon^{N+1} \mathbb{E}_x \left[\int_0^{\tau \wedge t} h(X_u) \eta_N(X_u, t-u) \, du \right]. \tag{22}$$

Proof. Please refer to the proof of Proposition 3.3.1, pp. 35-37 of [33]. □

Remark 2.2. *The error function* (22) *relies on the L^1-condition given by* (21)*. Our practices later show that the L^1-condition is easy to be justified. On the other hand, by referring to the proof in [33], we can also derive an operator representation of Eq.* (22)*. But the corresponding Green's function is for the operator $\beta - \mathcal{A}$ and given which it is very difficult to find a unified sufficient condition satisfying* (21)*.*

3. Applications

In this section, we use five diffusion processes to illustrate the applications of our perturbation framework. In the examples below, some of the closed-form density functions may already be found, but due to the complexity of the FPT problem, those densities may either be valid only for special cases (e.g. the OU FPTD under special case $\theta = 0$ [20]) or suffer computational efficiency issues (e.g. Bessel FPTDs in [21]). For these well-studied diffusion processes, in this section and the following numerical illustration part, we will show that the perturbation could provide accurate density asymptotics while maintain a much faster computing speed. On the other hand, considering the fact that there are still many diffusion processes whose closed-form FPTDs have not been found yet, therefore, another more important purpose of this section is to demonstrate that the perturbation framework can be applied to a wide branch of diffusion processes.

In order to keep a concise paper, in the contents below we only provide the perturbed density functions and comment on wherever necessary. More details of discussions and proofs of the results can be found in [33, Chapters 4-6] or the appendix of this paper.

3.1. Ornstein-Uhlenbeck Process

The OU process was first introduced to describe the velocity of a particle that follows a Brownian motion movement [52]. Later the process appears widely in neural science [32, 54] and mathematical finance [23, 28, 48, 53]. The FPT of the OU process is studied extensively (see [40] for a brief review), and the LT of the FPT can be found in [9, Eq. 2.0.1, p. 542]. Based on our settings, the drift function of the OU process is given by

$$h(x) = \theta - x,\ \theta \in \mathbb{R},\ x \in \mathbb{R}. \tag{23}$$

Refer to [52, 55], the OU process has a unique strong solution and it is recurrent in $\mathbb{R}$. Our framework therefore can be applied. For $a = 0$ and $x > 0$, we consider the hitting from above problem, i.e. $I = (0, +\infty)$.

Proposition 3.1. (N-th Order Perturbed FPTD of the OU Process.) *Let $N \in \mathbb{N}$, the N-th order perturbed downward FPTD of the OU process is given by*

$$p_\tau^{(N)}(t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{2\pi}} \sum_{n=0}^{2N-1} h_n t^{\frac{n}{2}-1} D_{-n+1}\left(\frac{x}{\sqrt{t}}\right),$$

where $D_\cdot(\cdot)$ is the parabolic cylinder function,

$$h_n := \sum_{\substack{i,j,k;\\ 2i-j-k=n}} \epsilon^i c_k^{(i,j)} \theta^k x^j,$$

and $\{c_k^{(i,j)}\}$ is a triple-indexed real sequence:

$$\{c_k^{(i,j)} : 0 \le i \le N,\ 0 \le j \le 2i,\ 0 \le k \le (2i-j) \wedge i\}$$

with its recursion representation given in Appendix A.

Proof. Please refer to [33, Proposition 4.1.3, p. 46]. □

Remark 3.1. *Recall in Proposition 2.1, a sufficient condition for the perturbation to converge is that h should be bounded in I. However, refer to* (23)*, apparently, this is not the case. In fact, according to [33, Proposition 4.1.4, p. 47], the right-tail asymptotic of the perturbed FPTD is $t^{N-\frac{3}{2}}$. Therefore, when $N \geq 2$ the perturbed density diverges when $t \uparrow +\infty$.*

Remark 3.2. *On the other hand, when $N = 1$, the right-tail asymptotic converges to 0 at the rate of $t^{-\frac{1}{2}}$. In this case, even though h is not bounded, by using Proposition 2.2, we have shown in [33, Proposition 4.2.2, p. 52] that the perturbed density converges in $t \in [0, T]$ for any $T > 0$ at the rate of $O(\epsilon^2)$. Moreover, the error representation in Eq.* (22) *is valid and the error function $\eta_1(x, t)$ is given by [33, Lemma 4.1.5, p. 49].*

Note that, the generalised case $a < x$ and hitting from below problem ($I = (-\infty, a)$) can be retrieved by using the affine transformation and the reflecting process, respectively.

Back to Proposition 3.1, in view of the close relation between the D-functions and the Hermite polynomials [35, Sections 12.7.1, 12.7.2], given $N = 1$, we can further write $p_\tau^{(N)}(t)$ explicitly in the following way:

$$p_\tau^{(1)}(t) = \left(1 + \epsilon(\frac{1}{2}x^2 - \theta x + \frac{1}{2}t)\right) \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}. \tag{24}$$

3.2. Bessel Process with $0 < n < 2$

Bessel process was introduced in [36] as the norm of a n-dimensional Brownian motion. Denoted by $BES(n)$ (sometimes by $BES(\nu)$ with $\nu = \frac{n-2}{2}$), the basic properties of the process have been discussed in [44, Chapter XI]. In mathematical finance, the family of Bessel processes is closely related to models of short rates and stochastic volatilities [10, 11, 12, 15, 23]. Similar to the OU process, the FPT of the Bessel process has been studied extensively. One can find the LTs for $n \geq 2$ in [9]; in [21], the LTs of $0 < n < 2$ (and other general n) and some explicit FPTDs are given.

We consider the class of Bessel processes with orders $n = 1 + \epsilon$ and $-1 < \epsilon < 1$. For $BES(1+\epsilon)$, the h-function is specified by

$$h(x) = \frac{1}{2x}, \ x \in (0, +\infty). \tag{25}$$

Let $I = (a, +\infty)$ with $0 < a < x$. Refer to [29] (also see [21, Eq. (2.5)]), the initial LT is given by the ratio of modified Bessel functions of the second kind. In fact, according to [21, Eq. (2), Theorem 2.2], the explicit FPTD of the downward hitting is given by (note that $\nu < 0$ in our case)

$$\begin{aligned} p_\tau(t) = &\left(1 - (\frac{a}{x})^{\frac{\epsilon-1}{2}} \int_0^{+\infty} \frac{L_{\frac{1-\epsilon}{2}, \frac{x}{a}}(y)}{y} e^{-\frac{y(x-a)}{a}} dy\right) \frac{x-a}{\sqrt{2\pi t^3}} e^{-\frac{(x-a)^2}{2t}} \\ &+ (\frac{a}{x})^{\frac{\epsilon-1}{2}} \int_0^t \frac{x-a}{\sqrt{2\pi s^3}} e^{-\frac{(x-a)^2}{2s}} \int_0^{+\infty} L_{\frac{1-\epsilon}{2}, \frac{x}{a}}(y) e^{-\frac{y(x-a)\sqrt{t}}{a\sqrt{s}}} \frac{(x-a)}{2a\sqrt{st}} dy dt, \end{aligned} \tag{26}$$

where

$$L_{\mu,c}(y) = \frac{\cos(\pi\mu)\left(I_\mu(cy)K_\mu(y) - I_\mu(y)K_\mu(cy)\right)}{K_\mu^2(y) + \pi^2 I_\mu^2(y) + 2\pi\sin(\pi\mu)K_\mu(y)I_\mu(y)},$$

and $I.(\cdot)$, $K.(\cdot)$ are the modified Bessel functions.

From (26) above we can see that the density involves a convolution of Bessel functionals. Therefore, the computing speed might be limited in practice (assuming there is no M.C. simulation/importance sampling or parallel computation used).

As an alternative, we provide the perturbed FPTD in the next proposition. Note that, in the case $I = (a, +\infty)$, the drift function in (25) and its derivative are bounded on I. According to Proposition 2.1, the perturbation series converges. But for the simplicity of calculations (actually, our numerical tests show the first order perturbation is accurate enough in providing density asymptotics), the result in the following proposition is only calculated up to the first order. Higher order results are left in the future work.

Proposition 3.2. (First Order Perturbed FPTD of $BES(1+\epsilon)$.) *The first order perturbed downward FPTD of $BES(1+\epsilon)$ is given by*

$$p_\tau^{(1)}(t) = \Big(1 + \frac{\epsilon}{2}\ln(\frac{a}{x})\Big)\frac{x-a}{\sqrt{2\pi t^3}}e^{-\frac{(x-a)^2}{2t}} + \frac{\epsilon(x-a)}{\sqrt{2\pi t}}\int_{(x-a)^2}^{\infty}\frac{1}{2t(\sqrt{y}-x+3a)(\sqrt{y}+x+a)}e^{-\frac{y}{2t}}\,dy. \tag{27}$$

Proof. Please refer to the proof of Proposition 5.1.3, pp. 72-73 of [33]. □

Remark 3.3. *The perturbed FPTD* (27) *maintains the inverse Gamma part as in the original density function (Eq.* (26)*). From the proposition above, we can clearly see that the first order truncation indeed is a simplification to higher orders and which originally are represented using the Bessel functionals. In terms of the truncation error, we have shown in [33, Proposition 5.2.2, p. 77] that $p_\tau^{(1)}$ converges at the rate of $O(\epsilon^2)$, and the error function $\eta_1(x,t)$ can be found therein.*

3.3. Exponential-Shiryaev Process

In this subsection, we consider a newly introduced three-parameter diffusion process:

$$dX_t = \epsilon\left(e^{-2\alpha X_t} - c\right)dt + dW_t,\ X_0 = x, \tag{28}$$

with $\epsilon \geq 0$, $\alpha \geq 0$, $0 \leq c \leq 1$, and $x \in \mathbb{R}$. The process is occasionally found by us during this research. The motivation of studying such a process is due to its special drift function properties. Note that the exponential function is closed under differentiation and integration, therefore, by applying the perturbation mechanism we expected a neat mathematical function form in the perturbed density. We have conducted fundamental analysis of the process and realised that the sample path of $\{X_t\}_{t\geq 0}$ can be used to describe the log-price of economic bubbles (cf. [13] and [33, Chapter 6, p. 87]).

The Shiryaev process [49, 50] refers to the following SDE which was derived by A.N. Shiryaev in the context of sequential analysis:

$$dZ_t = (1 + \mu Z_t)dt + \sigma dW_t,\ \mu \in \mathbb{R},\ \sigma > 0.$$

Later, in a paper of G. Peskir [41], he has named the process above to be the Shiryaev process and analysed the transition density of the process. The transition density of the Shiryaev process is linked to the Hartman-Watson distribution [22] and which is further involved in the Asian-option pricing problem. The reason being of naming SDE (28) as the exponential-Shiryaev process is due to the fact that the exponential transform of $\{X_t\}_{t\geq 0}$

$$Y_t := e^{2\alpha X_t}$$

is a scaled version of $\{Z_t\}_{t\geq 0}$, i.e.

$$Y_t = 2\alpha\epsilon Z_t \tag{29}$$

with $\mu = \frac{\alpha}{\epsilon} - c$ and $\sigma = \frac{1}{\epsilon}$.

Our present paper focuses on illustrating the perturbed FPTD of the exponential-Shiryaev process (SDE (28)) (in fact, based on (29) and using the FPTD of $\{X_t\}_{t\geq 0}$, the FPTD of the Shiryaev process can be found accordingly). We have shown in [33, Chapter 6, p. 87] that $\{X_t\}_{t\geq 0}$ has a unique strong solution, is strong Markov, and its FPT is finite *a.s.* The unique solution of the BVP therefore exists. Refer to [33, Section 6.4.1, pp. 105-107], for a hitting level $a \in \mathbb{R}$, the original LTs are given by

$$f(x,\beta) = \begin{cases} e^{\lambda(x)-\lambda(a)} M\left(m,n,\psi(x)\right)/M\left(m,n,\psi(a)\right),\ x > a \\ e^{\lambda(x)-\lambda(a)} U(m,n,\psi(x))/U(m,n,\hat{\psi}(a)),\ x < a \end{cases}, \tag{30}$$

where $M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ are the Kummer's functions and

$$\begin{cases} m := (\sqrt{c^2\epsilon^2 + 2\beta} - \epsilon c)/(2\alpha) \\ n := (\sqrt{c^2\epsilon^2 + 2\beta} + \alpha)/\alpha \\ \psi(x) := \epsilon e^{-2\alpha x}/\alpha \\ \lambda(x) := -2\alpha m x \end{cases}.$$

For a quick illustration of the perturbation, we consider the hitting from above problem. *W.l.o.g.*, we let $a = 0$ (general $a \in \mathbb{R}$ can be solved by using the affine transformation) and $I = (0, +\infty)$. According to Proposition 2.1, the drift function

$$h(x) = e^{-2\alpha x} - c,\ x \in I$$

and its derivative are bounded and therefore the perturbation series converges. Similar as in the previous subsection, in the proposition below we provide the first order truncated result. Under a special case $c = 0$, a recursion solution of higher order ODEs and the inverse transforms can be found in [33, Appendix 6.A, pp. 124-127].

Proposition 3.3. (First Order Perturbed FPTD of the Exponential-Shiryaev Process.) *The first order perturbed downward FPTD of* $\{X_t\}_{t\geq 0}$ *is given by*

$$\begin{aligned} p_\tau^{(1)}(t) = &\left(1 + \epsilon(cx + \frac{\left(1 - e^{-2\alpha x}\right)(\alpha t - x)}{2\alpha x})\right) \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \\ &- \epsilon \frac{\alpha}{4}\left(1 - e^{-2\alpha x}\right) e^{\alpha x\left(\frac{\alpha t}{2x}+1\right)} Erfc\left(\frac{x}{\sqrt{2t}} + \alpha\sqrt{\frac{t}{2}}\right), \end{aligned} \tag{31}$$

where $Erfc(\cdot)$ *is the* complementary error function*.*

Proof. Please refer to the proof of Proposition 6.4.1, pp. 108-109 in [33]. □

3.4. Hyperbolic Ornstein-Uhlenbeck Process

In the following two subsections, we consider two special processes from the hypergeometric diffusion class [8]. The name of the 'hypergeometric diffusion' comes from the fact that the LTs of their FPTs to a constant level are connected to the hypergeometric functions. Refer to [8, Eq. (4.1)], the hypergeometric diffusion is defined as

$$dZ_t = \left((\nu + \frac{1}{2})c\coth(cZ_t) - \frac{\rho}{c}\tanh(cZ_t)\right)dt + dW_t,\ Z_0 > 0, \tag{32}$$

with $\nu \geq -\frac{1}{2},\ \rho \geq 0,\ c \geq 0$.

Note that, the hypergeometric diffusion can be regarded as generalisations to other familiar stochastic processes. For example, let $c \downarrow 0$ with $\nu \neq -\frac{1}{2}$ and $\rho \neq 0$, we then have the radial OU process. Also, keep $c \downarrow 0$, $\nu \neq -\frac{1}{2}$, but $\rho = 0$, we get the Bessel process. Similarly, for $c \downarrow 0$, $\rho \neq 0$, but $\nu = -\frac{1}{2}$, the OU process is acquired.

In this subsection, we focus on the case of $c > 0$ and $\nu = -\frac{1}{2}$. Let $\rho = \epsilon > 0$ and further consider $\theta \in \mathbb{R}$, we define the hyperbolic OU process as follows

$$dX_t = \epsilon(\theta - \frac{\tanh(cX_t)}{c})dt + dW_t,\ X_0 \in \mathbb{R}. \tag{33}$$

The drift function of $\{X_t\}_{t\geq 0}$ is correspondingly given by

$$h(x) = \theta - \frac{\tanh(cx)}{c}, \; x \in \mathbb{R}. \tag{34}$$

Again, for the simplicity of presentation, we demonstrate only the hitting from above case. The hitting from below case can be retrieved by using the symmetry of $\tanh(\cdot)$. Let $a > 0$ be the hitting level and $I = (a, +\infty)$, after tedious calculations, we have the following LT of the true FPTD (solution to the original BVP problem):

$$f(x,\beta) = \frac{L(x,\beta)}{L(a,\beta)}, \tag{35}$$

where

$$\begin{aligned} L(x,\beta) :=& \frac{(\tanh(cx)-1)^{\frac{1}{2}(A_1+A_2+\epsilon(1+\theta c)/c^2)}}{(\tanh(cx)+1)^{\frac{\epsilon\,\theta\,c+c^2+\epsilon}{2c^2}}} \Bigg(\\ & {}_2\mathrm{F}_1(A_1,B_1;C_1;\,1)\,{}_2\mathrm{F}_1(A_2,B_2;C_2;Z(x))\,(2\,\tanh(cx)+2)^{\frac{1}{2}C_2}\,4^{C_1-1} \\ & -{}_2\mathrm{F}_1(A_2,B_2;C_2;\,1)\,{}_2\mathrm{F}_1(A_1,B_1;C_1;Z(x))\,(2\,\tanh(cx)+2)^{\frac{1}{2}C_1}\Bigg), \end{aligned}$$

with

$$\begin{cases} A_1 := \frac{\sqrt{(c\theta-1)^2\epsilon^2+2\,\beta\,c^2}-2\,\epsilon+\sqrt{(c\theta+1)^2\epsilon^2+2\,\beta\,c^2}}{2c^2} \\ A_2 := \frac{\sqrt{(c\theta-1)^2\epsilon^2+2\,\beta\,c^2}-2\,\epsilon-\sqrt{(c\theta+1)^2\epsilon^2+2\,\beta\,c^2}}{2c^2} \\ B_1 := 1+\frac{2\epsilon}{c^2}+A_1 \\ B_2 := 1+\frac{2\epsilon}{c^2}+A_2 \\ C_1 := 1+\frac{\sqrt{(c\theta+1)^2\epsilon^2+2\,\beta\,c^2}}{c^2} \\ C_2 := 2-C_1 \\ Z(x) := \frac{1}{2}\tanh(cx)+\frac{1}{2} \end{cases}$$

and ${}_2\mathrm{F}_1(\cdot,\cdot;\cdot;\cdot)$ to be the Gauss hypergeometric function.

The LT in (35) is different to which is shown in [8, Section 6]. The latter one in fact only gives the LT of the reflecting hyperbolic OU process with a special case $\theta = 0$. From those equations above, we can see that the LT of the hyperbolic OU FPTD is

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The case $a < 0$ can be solved as well. However, it involves different series representation in the perturbed density. For the simplicity of presentation we skip the negative part, but the result can be provided upon request.

already complicated enough. It can be imagined that finding the explicit inverse would be even more difficult or not even be possible. Therefore, we provide the first order perturbed FPTD below as an asymptotic to the original density (note that, the drift function in (34) and its derivative are bounded in $\mathbb{R}$, and according to Proposition 2.1 the perturbed series converges).

Proposition 3.4. (First Order Perturbed FPTD of the Hyperbolic OU Process.) *The first order perturbed downward FPTD of the hyperbolic OU process is given by*

$$p_\tau^{(1)}(t) = \left(1 + \epsilon(x-a)(\frac{1}{c} - \theta)\right) \frac{x-a}{\sqrt{2\pi t^3}} e^{-\frac{(x-a)^2}{2t}} - \frac{\epsilon}{c} T(t), \tag{36}$$

where

$$\begin{aligned} T(t) :=& (A(x) - A(a)) \sqrt{\frac{2}{\pi}} t^{-\frac{5}{2}} \left((x-a)^2 - t\right) e^{-\frac{(x-a)^2}{2t}} \\ &+ \sum_{n=1}^{\infty} \Bigg[(B(x,n) - B(a,n)) \Bigg(\frac{1}{\sqrt{2\pi}} \left(\frac{(x-a)t^{-\frac{3}{2}}}{2} - cnt^{-\frac{1}{2}} \right) e^{-\frac{(x-a)^2}{2t}} \\ &+ \frac{c^2 n^2}{2} e^{cn(x-a) + \frac{c^2 n^2 t}{2}} \mathit{Erfc}\left(\frac{(x-a)}{\sqrt{2t}} + \frac{cn}{\sqrt{2}} \sqrt{t} \right) \Bigg) \\ &- (C(x,n) - C(a,n)) \Bigg(\sqrt{\frac{2}{\pi}} t^{-\frac{3}{2}} \left(\frac{(x-a)^2}{t} - 1 - cn(x-a) + c^2 n^2 t \right) e^{-\frac{(x-a)^2}{2t}} \\ &- c^3 n^3 e^{cn(x-a) + \frac{c^2 n^2 t}{2}} \mathit{Erfc}\left(\frac{(x-a)}{\sqrt{2t}} + \frac{cn}{\sqrt{2}} \sqrt{t} \right) \Bigg) \Bigg], \end{aligned}$$

given that

$$\begin{cases} A(x) := \frac{2\mathrm{e}^{cx} \arctan(\mathrm{e}^{cx})}{c^2} - \frac{\ln\left((\mathrm{e}^{cx})^2 + 1\right)}{c^2} - \frac{\pi e^{cx}}{c^2} + \frac{2x}{c} \\ B(x,n) := (-1)^n \frac{e^{-2ncx}}{cn(2n+1)} \\ C(x,n) := (-1)^n \frac{e^{-2ncx}}{c^2 n(2n+1)} \end{cases}.$$

Proof. Please refer to Appendix B. □

Remark 3.4. *Refer to the proof in Appendix B, we can find that the series representation of the perturbed density is from the expansion of the* arctan*-function (Eqs. (44) and (45)). In fact, the series converges very fast, especially by noticing that [39,* *7.1.23, p. 298]*

$$e^{cn(x-a)+\frac{c^2n^2t}{2}}\,Erfc\left(\frac{(x-a)}{\sqrt{2t}}+\frac{cn}{\sqrt{2}}\sqrt{t}\right) \sim e^{cn(x-a)+\frac{c^2n^2t}{2}}\frac{e^{-(\frac{(x-a)}{\sqrt{2t}}+\frac{cn}{\sqrt{2}}\sqrt{t})^2}}{(\frac{(x-a)}{\sqrt{2t}}+\frac{cn}{\sqrt{2}}\sqrt{t})\sqrt{\pi}}$$
$$=\frac{e^{-\frac{(x-a)^2}{2t}}}{(\frac{(x-a)}{\sqrt{2t}}+\frac{cn}{\sqrt{2}}\sqrt{t})\sqrt{\pi}},\ n\uparrow+\infty.$$

3.5. Hyperbolic Bessel Process

Recall (32), by letting $\nu=-\frac{1}{2}+\epsilon$ with $\epsilon>0$ we have the hyperbolic Bessel process:

$$dX_t=\epsilon c\coth(cX_t)dt+dW_t,\ X_0>0. \tag{37}$$

From [9, No. 33, p. 70], it can be seen that the hyperbolic Bessel process is used in the path decomposition of the drifted Brownian motion (with a non-negative drift) conditioning on not hitting 0. More detailed discussions and transition densities of the hyperbolic Bessel process can be found in [8, Section 5].

In this subsection, we consider the hitting from above case. Let $a>0$ and $I=(a,+\infty)$. The h-function of $\{X_t\}_{t\geq 0}$ is given by

$$h(x)=c\coth(cx),\ x>0 \tag{38}$$

and which together with its derivative are bounded in I. Again, by Proposition 2.1 we know that the perturbation series converges.

Refer to [8, Eqs. (1.3), (4.3), (4.4)], the original LT of the downward FPTD is given by

$$f(x,\beta)=\frac{\cosh^{2\alpha}(ca)_2\mathrm{F}_1(\alpha,\alpha+\frac{1}{2};2\alpha-\epsilon+1;\,1-\tanh^2(cx))}{\cosh^{2\alpha}(cx)_2\mathrm{F}_1(\alpha,\alpha+\frac{1}{2};2\alpha-\epsilon+1;\,1-\tanh^2(ca))}, \tag{39}$$

where

$$\alpha=\frac{\sqrt{\epsilon^2+\frac{2\beta}{c^2}}+\epsilon}{2}.$$

Similar to the hyperbolic OU FPTD, regarding the complex structure of the LT, it seems to be not an easy task of finding the explicit inverse of (39). In the next proposition, as an alternative, we provide the first order perturbed FPTD.

Proposition 3.5. (First Order Perturbed FPTD of the Hyperbolic Bessel Process.) *The first order perturbed downward FPTD of the hyperbolic Bessel process is given by*

$$p_\tau^{(1)}(t)=(1-\epsilon c(x-a))\,\frac{x-a}{\sqrt{2\pi t^3}}e^{-\frac{(x-a)^2}{2t}}+\epsilon\tilde{T}(t), \tag{40}$$

where

$$\begin{aligned}\tilde{T}(t) :=& (\tilde{A}(x) - \tilde{A}(a))\sqrt{\frac{2}{\pi}}t^{-\frac{5}{2}}\left((x-a)^2 - t\right)e^{-\frac{(x-a)^2}{2t}} \\ &+ \sum_{n=1}^{\infty}\Bigg[(\tilde{B}(x,n) - \tilde{B}(a,n))\Bigg(\frac{1}{\sqrt{2\pi}}\left(\frac{(x-a)t^{-\frac{3}{2}}}{2} - cnt^{-\frac{1}{2}}\right)e^{-\frac{(x-a)^2}{2t}} \\ &+ \frac{c^2n^2}{2}e^{cn(x-a)+\frac{c^2n^2t}{2}}\,Erfc\left(\frac{(x-a)}{\sqrt{2t}} + \frac{cn}{\sqrt{2}}\sqrt{t}\right)\Bigg) \\ &- (\tilde{C}(x,n) - \tilde{C}(a,n))\Bigg(\sqrt{\frac{2}{\pi}}t^{-\frac{3}{2}}\left(\frac{(x-a)^2}{t} - 1 - cn(x-a) + c^2n^2t\right)e^{-\frac{(x-a)^2}{2t}} \\ &- c^3n^3e^{cn(x-a)+\frac{c^2n^2t}{2}}\,Erfc\left(\frac{(x-a)}{\sqrt{2t}} + \frac{cn}{\sqrt{2}}\sqrt{t}\right)\Bigg)\Bigg],\end{aligned}$$

and

$$\begin{cases}\tilde{A}(x) := -\frac{1-e^{cx}}{c}\ln(1-e^{-cx}) - \frac{1+e^{cx}}{c}\ln(1+e^{-cx}) \\ \tilde{B}(x,n) := \frac{e^{-2ncx}}{n(2n+1)} \\ \tilde{C}(x,n) := \frac{e^{-2ncx}}{nc(2n+1)}\end{cases}.$$

Proof. Please refer to Appendix B. □

Remark 3.5. *Those two T-functions in the above proposition and Proposition 3.4 are very similar (the convergence in Proposition 3.5 is guaranteed by referring to Remark 3.4), apart from that the series functions A, B, C and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are different. In fact, the series functions are representations of integrals involving* ln- *(cf. Eqs.* (52) *and* (53)*) and* arctan*-functions respectively, while the inverse LT in those two problems are the same. Instead of using the series representation, one can also refer to the proof in Appendix B to derive an integral form. We leave this to the reader.*

Remark 3.6. *Propositions 3.4 and 3.5 also provide building blocks for the perturbed FPTD of the hyperbolic radial OU process (*Hyp-ROU*). If we define h_1 and h_2 to be the drift functions (cf. Eqs.* (34), (38)*) of the hyperbolic OU (*Hyp-OU*) and the hyperbolic Bessel (*Hyp-Bes*) processes respectively, then the h-function of the Hyp-ROU is given by $h = h_1 + h_2$. Refer to the recursion formula in Remark 2.1, the first order perturbed LT of the Hyp-ROU is*

$$f_1^{Hyp-ROU}(x,\beta) = f_1^{Hyp-OU}(x,\beta) + f_1^{Hyp-Bes}(x,\beta),$$

where f_1^{Hyp-OU}, $f_1^{Hyp-Bes}$ are the first order perturbed LTs of the Hyp-OU and the Hyp-Bes (cf. Eqs. (49) *and* (54)*). Then by re-organising the proofs in Appendix B,* *the first order perturbed FPTD of the Hyp-ROU is given by*

$$p_\tau^{(1),\ Hyp-OU} = \left(1 + \epsilon(x-a)(\frac{1}{c} - \theta - c)\right) \frac{x-a}{\sqrt{2\pi t^3}} e^{-\frac{(x-a)^2}{2t}} + \epsilon\left(\tilde{T}(t) - \frac{T(t)}{c}\right).$$

*Similar result can be acquired for the radial OU process (*ROU, *cf. Sections 3.1 and 3.2):*

$$p_\tau^{(1),\ ROU}(t) = \left(1 + \epsilon(\frac{1}{2}(x-a)^2 - \theta(x-a) + \frac{1}{2}t + \frac{1}{2}\ln(\frac{a}{x}))\right) \frac{x-a}{\sqrt{2\pi t^3}} e^{-\frac{(x-a)^2}{2t}}$$
$$+ \frac{\epsilon(x-a)}{\sqrt{2\pi t}} \int_{(x-a)^2}^{\infty} \frac{1}{2t(\sqrt{y} - x + 3a)(\sqrt{y} + x + a)} e^{-\frac{y}{2t}} dy.$$

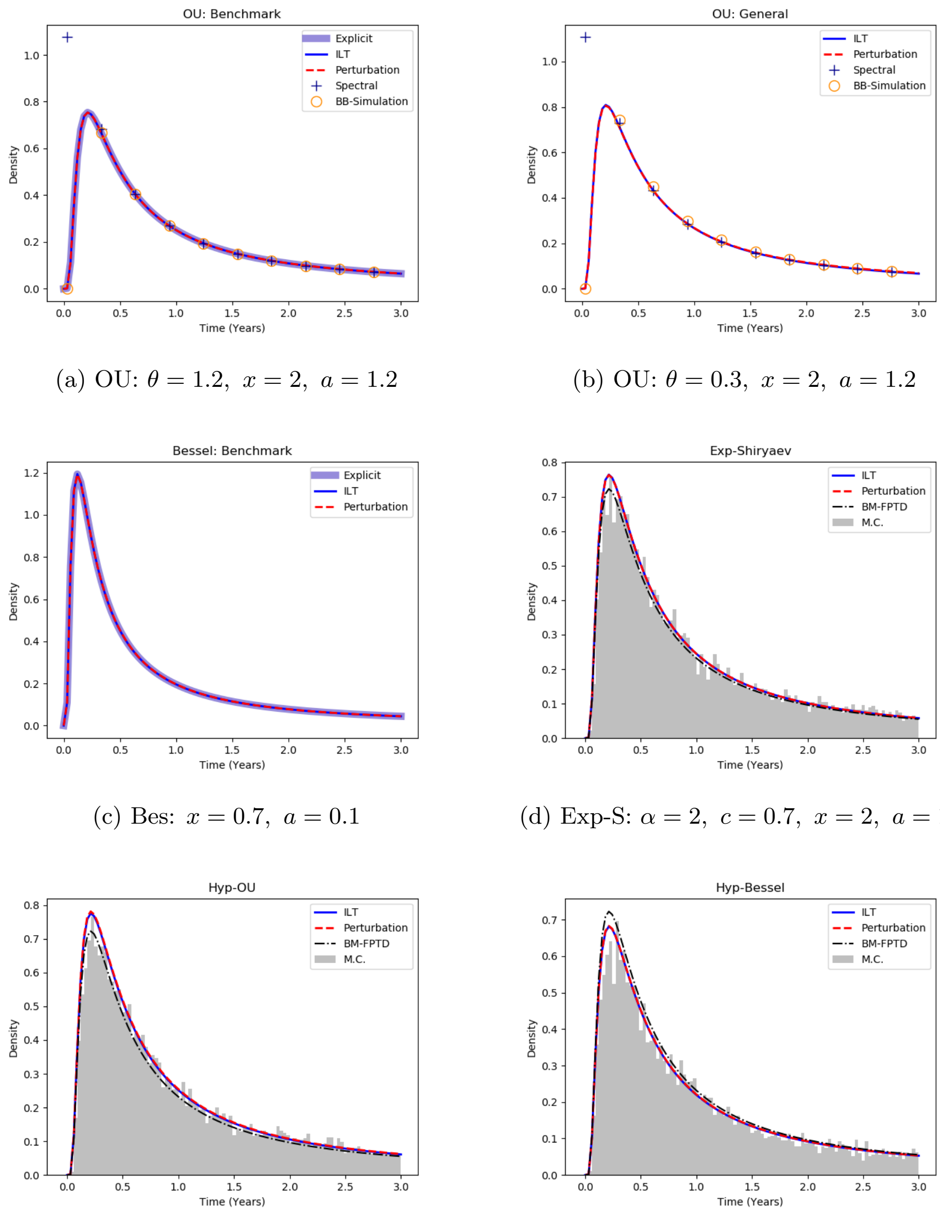
4. Numerical Examples

In this section, we demonstrate 6 numerical examples (Figure 1) of the comparisons between the first order perturbed FPTDs (cf. Propositions 3.1-3.5) and other well-studied methods. The perturbation parameter in all examples is chosen to be $\epsilon = 0.1$, and other parameters of different diffusions are listed in captions under Figures 1a-1f. Computing times of different methods are reported in Table 1 and deviation statistics of each method to the true values are reported in Table 2.

Among those 6 figures, two of them (Figures 1a and 1b) illustrate the FPTDs of the OU process. In Figure 1a, we consider a special case $\theta = a$ and the explicit density formula is given in [20]; the comparison is made among the perturbation/inverse LT (*ILT*, cf. the Talbot approach in [1]), the explicit density, spectral decomposition, and 3-dimensional Brownian bridge simulation (BB-simulation) approaches (cf. [2] and [25], respectively). Figure 1b presents a general scenario where $\theta \neq a$. Note that in this case there is no closed-form FPTD found yet.

In Figure 1c, the Bessel process with a non-integer order $n = 1.1$ is studied. Different to other examples, the starting point and the hitting level are chosen as $x = 0.7$ and $a = 0.1$. The reason of not selecting large levels (e.g. $x = 2$ and $a = 1.2$) is because that the Bessel process behaves very akin to the Brownian motion when $X_t >> 0$. In this example, we include the explicit density function (cf. Eq. (26)) as the benchmark for the perturbation and the Talbot inverse algorithm.

Figures 1d to 1f are similar in their presentation formats. We consider respectively the FPTDs of the exponential-Shiryaev, the hyperbolic-OU, and the hyperbolic-Bessel



(a) OU: $\theta = 1.2,\ x = 2,\ a = 1.2$

(b) OU: $\theta = 0.3,\ x = 2,\ a = 1.2$

(c) Bes: $x = 0.7,\ a = 0.1$

(d) Exp-S: $\alpha = 2,\ c = 0.7,\ x = 2,\ a = 1.2$

(e) Hyp-OU: $c = 0.5,\ \theta = 0.5,\ x = 2,\ a = 1.2$

(f) Hyp-Bes: $c = 0.5,\ x = 2,\ a = 1.2$

FIGURE 1: First order perturbed FPTDs and comparisons with $\epsilon = 0.1$

processes. Apart from providing their perturbed and ILT FPTDs, we also include the

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The ILT in Figure 1d is from the Talbot algorithm while computations in Figures 1e and 1f are

	Perturbation	ILT	M.C.	Explicit	Spectral*	BB-Sim*
OU Benchmark	9.16e-4	1.08	*n.a.*	1.08e-3	*n.a.*	*n.a.*
OU General	9.53e-4	1.74	*n.a.*	*n.a.*	3.27e3	1.16e3
Bessel	0.10	0.03	*n.a.*	3.04e2	*n.a.*	*n.a.*
Exp-Shiryaev	2.46e-3	0.18	3.38	*n.a.*	*n.a.*	*n.a.*
Hyp-OU	0.09	0.15	3.53	*n.a.*	*n.a.*	*n.a.*
Hyp-Bessel	0.08	0.13	3.48	*n.a.*	*n.a.*	*n.a.*

Table 1: Computing times (seconds) of density curves in Figure 1

brute-force Monte Carlo simulation (*M.C.*) and the standard Brownian motion (*BM*) FPTDs. Note that, the black dashed curves in the last three graphs are the same since the BM FPTD is only determined by the starting point and the hitting level.

The results from Figure 1 show that the first order perturbation works well for all diffusion processes discussed above. Also, Figures 1d-1f indicate that, even for small ϵ, one cannot use the BM FPTD to acquire the desired approximation to the FPTDs of structurally different diffusions (although we do not provide the graphs for the OU and the Bessel processes, this is also the case); the M.C. approach cannot give accurate estimations either. On the other hand, by checking the computing times in Table 1 (note that, times of Spectral* and BB-Sim* are scaling-up estimates from those 10 density points in Figure 1b), one may realise that the perturbation possesses the fastest computing speed in almost all scenarios (apart from that is is slower than the ILT in the Bessel FPTD).

In Table 2, we further compare deviations of each methods to the true values (explicit solutions wherever applicable). All statistics are calculated in absolute manners but note that for the spectral decomposition and the BB-simulation, there are only 10 density points (v.s. 100 density points in other methods). The table shows that the ILT produces more accurate results than our perturbation method in both processes. The average deviation shows that the errors of the perturbation are at the magnitude of 10e-4 though, at the total variation level, the perturbation generates about 3%-

from the Gaver-Stehfest algorithm [1]. The change in those two hypergeometric diffusions is due to the restriction of Python in computing the hypergeometric functions with complex numbers.

		Perturbation	ILT	Spectral*	BB-Sim*
	Total-Deviation	0.03	3.76e-3	1.09	3.29e-4
OU Benchmark	Max-Deviation	7.02e-4	3.67e-4	1.07	6.03e-5
(Fig. 1a)	Avg.-Deviation	3.49e-4	3.80e-5	0.11	3.29e-5
	Total-Deviation	0.05	3.04e-3	*n.a.*	*n.a.*
Bes Benchmark	Max-Deviation	2.35e-3	4.69e-4	*n.a.*	*n.a.*
(Fig. 1c)	Avg.-Deviation	5.09e-4	3.07e-5	*n.a.*	*n.a.*

TABLE 2: Statistics of absolute deviations in Figure 1

5% error. However, considering that the purpose of the current paper is to produce the density curve instead of the probability curve, we would not consider the total deviation here as a major benchmark. On the other hand, as can be found in [2], the spectral decomposition has a convergence issue when t is small and this explains its large deviations we observed in Table 2. The BB-simulation produces the smallest error among four methods, but by referring to Table 1 we see that it is based on the cost of an extremely slow computation speed.

Another potential question is about the error behaviour on different model parameters. In [33, Chapters 4, 5, 6], we have demonstrated perturbation error functions for different diffusion processes. But unfortunately, a uniform error bound has not been found yet. In general, by referring to Proposition 2.2 we know for sure that the smaller the ϵ, the smaller the error. Apart from tuning ϵ, based on observations we also found that smaller t and smaller ratios of x/a would also generate smaller errors. Taking the benchmark OU process (benchmark cases where $\theta = a = 1.2$) as an example, by choosing larger ϵ, smaller ratio of x/a, smaller t ($\epsilon = 1.2,\ x = 1.3,\ t \leq 1$), the average deviation reported in Table 2 becomes 7.00e-3; and by choosing smaller ϵ, larger ratio of x/a, larger t ($\epsilon = 0.3,\ x = 9,\ t \leq 5$), the average deviation increases to 3.26e-2. Apparently there is a complicated relation among ϵ, x, θ, and a. But to understand it better a more serious study should be conducted. We leave this in the future work.

As a last remark of this section, we highlight a few findings in higher order OU

In principle, the probability curve can be retrieved using the perturbation on different BVPs. We will leave this in future works.

		$N = 1$	$N = 3$	$N = 5$	$N = 7$	$N = 9$
$\epsilon = 0.1$	Avg.-Deviation	1.09e-3	5.78e-5	4.11e-6	3.35e-7	2.96e-8
	Large-t-Deviation	2.04e-3	2.31e-4	2.48e-5	2.68e-6	2.92e-7
$\epsilon = 0.3$	Avg.-Deviation	1.21e-2	6.52e-3	4.58e-3	3.58e-3	2.96e-3
	Large-t-Deviation	1.80e-2	2.25e-2	2.47e-2	2.60e-2	2.70e-2

Table 3: OU benchmark deviations with different perturbation orders

perturbations. Recall in Remark 3.1, we have mentioned that the higher order OU perturbations diverge on the right tail with $t^{N-\frac{3}{2}}$. But this does not necessarily mean that the higher order densities diverge for large t (at least not large enough to appear the divergence); in fact, depending on the choices of ϵ and other parameters, higher order densities may also produce more accurate results. Using parameters $x = 2$, $\theta = a = 1.2$ again, in Table 3 we report average deviations and large t deviations (t=10 years) for perturbed density curves compared to the true values given $\epsilon \in \{0.1, 0.3\}$. The time range is chosen to be $t \in [0, 10]$. Results show for $\epsilon = 0.1$, the perturbation converges up to t=10, but which slowly diverges for $\epsilon = 0.3$.

5. Conclusion

In this paper, we provide a systematic approach for solving the closed-form FPTD asymptotics of diffusion processes. We show a sufficient condition for the convergence of the perturbation series and derive a probabilistic representation for the error. The perturbation resulted closed-form solution does not only increase the computational efficiency, but also provides analytical tractability in understanding the FPTDs at extreme times. Using the framework we demonstrate valid approximations to FPTDs of various diffusion processes. Potential applications of this paper could be found in survival analysis, financial mathematics, and many others. Further works could be done in exploring the perturbed FPTDs of other diffusion processes, analysing error behaviours with different model parameters, and finding FPTD via simulations using the probabilistic representation in Proposition 2.2, etc.

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Appendix A. Recursive Structure in the OU Perturbation

Result A.1. (*Decomposition Structure I.*) For $i = 1$ and $i = 2$, $\left\{c_k^{(i,j)}\right\}$ is explicitly given by

$$i = 1 : \begin{cases} c_0^{(1,2)} = \frac{1}{2} \\ c_0^{(1,1)} = \frac{1}{2};\ c_1^{(1,1)} = -1 \end{cases}, \quad i = 2 : \begin{cases} c_0^{(2,4)} = \frac{1}{8} \\ c_0^{(2,3)} = \frac{1}{12};\ c_1^{(2,3)} = -\frac{1}{2} \\ c_0^{(2,2)} = -\frac{1}{8};\ c_1^{(2,2)} = 0;\ c_2^{(2,2)} = \frac{1}{2} \\ c_0^{(2,1)} = -\frac{1}{8};\ c_1^{(2,1)} = \frac{1}{2};\ c_2^{(2,1)} = -\frac{1}{2} \end{cases}.$$

Result A.2. (*Decomposition Structure II.*) For $i \geq 3$, $\left\{c_k^{(i,j)}\right\}$ is recursively determined by

$$
\begin{cases}
j = 2i: & c_0^{(i,2i)} = \frac{c_0^{(i-1,2i-2)}}{2i} \\[2ex]
j = 2i-1: & \begin{cases} c_0^{(i,2i-1)} = \frac{1}{(2i-1)} c_0^{(i-1,2i-3)} - \frac{2i-3}{2(2i-1)} c_0^{(i-1,2i-2)} \\ c_1^{(i,2i-1)} = \frac{1}{(2i-1)} \left(c_1^{(i-1,2i-3)} - c_0^{(i-1,2i-2)} \right) \end{cases} \\[2ex]
2 < j < 2i-1: & \begin{cases} k = (2i-j) \wedge i: & \begin{cases} j > i: \ c_{2i-j}^{(i,j)} = \frac{1}{j} \left(c_{2i-j}^{(i-1,j-2)} - c_{2i-j-1}^{(i-1,j-1)} \right) \\ j = i: \ c_i^{(i,i)} = -\frac{1}{i} c_{i-1}^{(i-1,i-1)} \\ j < i: \ c_i^{(i,j)} = \frac{1}{2}(j+1) c_i^{(i,j+1)} \\ \qquad\qquad -\frac{1}{j} c_{i-1}^{(i-1,j-1)} + c_{i-1}^{(i-1,j)} \end{cases} \\[2ex] 0 < k < (2i-j) \wedge i: & c_k^{(i,j)} = \frac{1}{2}(j+1) c_k^{(i,j+1)} + \frac{1}{j} c_k^{(i-1,j-2)} \\ & \qquad -\frac{j-1}{j} c_k^{(i-1,j-1)} - \frac{1}{j} c_{k-1}^{(i-1,j-1)} + c_{k-1}^{(i-1,j)} \\[2ex] k = 0: & c_0^{(i,j)} = \frac{1}{2}(j+1) c_0^{(i,j+1)} + \frac{1}{j} c_0^{(i-1,j-2)} - \frac{j-1}{j} c_0^{(i-1,j-1)} \end{cases} \\[2ex]
j = 2: & \begin{cases} k = i: & c_i^{(i,2)} = \frac{3}{2} c_i^{(i,3)} - \frac{1}{2} c_{i-1}^{(i-1,1)} + c_{i-1}^{(i-1,2)} \\ 0 < k < i: & c_k^{(i,2)} = \frac{3}{2} c_k^{(i,3)} - \frac{1}{2} c_k^{(i-1,1)} - \frac{1}{2} c_{k-1}^{(i-1,1)} + c_{k-1}^{(i-1,2)} \\ k = 0: & c_0^{(i,2)} = \frac{3}{2} c_0^{(i,3)} - \frac{1}{2} c_0^{(i-1,1)} \end{cases} \\[2ex]
j = 1: & \begin{cases} k = i: & c_i^{(i,1)} = c_i^{(i,2)} + c_{i-1}^{(i-1,1)} \\ 0 < k < i: & c_k^{(i,1)} = c_k^{(i,2)} + c_{k-1}^{(i-1,1)} \\ k = 0: & c_0^{(i,1)} = c_0^{(i,2)} \end{cases}
\end{cases}
.
$$

Appendix B. Proofs of Propositions 3.4 and 3.5

Proof. **Part I (Hyperbolic OU Perturbed LT).** We first consider a simplified version of (34). Let $h(x) = \tanh(cx)$ and denote by $\gamma := \sqrt{2\beta}$. According to the formula in Remark 2.1, the first order solution is

$$\tilde{f}_1 = f_0 \left[\int_a^x 2e^{2\gamma y} \left(\gamma \int_a^y \tanh(cz) e^{-2\gamma z} dz + C_1 \right) dy + C_2 \right]. \tag{41}$$

Rewrite

$$\gamma \int_a^y \tanh(cz) e^{-2\gamma z} dz = \gamma \int_a^y \tanh(cz) e^{-cz} e^{cz - 2\gamma z} dz, \tag{42}$$

and note that

$$\int \tanh(cz) e^{-cz} dz = -\frac{\sinh(cz)}{c} + 2\,\frac{\arctan(\mathrm{e}^{cz})}{c} + \frac{\cosh(cz)}{c}.$$

Then integral by parts of (42) yields

$$\begin{aligned}
&\gamma \int_a^y \tanh(cz) e^{-cz} e^{cz-2\gamma z} dz = -\gamma(c - 2\gamma) \int_a^y 2\,\frac{\arctan(\mathrm{e}^{cz})}{c} e^{cz-2\gamma z} dz \\
&+ \gamma \left(-\frac{\sinh(cz)}{c} + 2\,\frac{\arctan(\mathrm{e}^{cz})}{c} + \frac{\cosh(cz)}{c} \right) e^{cz-2\gamma z} \Big|_{z=a}^{y} \\
&- \gamma(c - 2\gamma) \frac{(-\cosh(cz) + \sinh(cz))\, \mathrm{e}^{cz - 2\,\gamma\, z}}{2\gamma\, c} \Big|_{z=a}^{y}.
\end{aligned} \tag{43}$$

The first term on the right-hand side needs special care. Note that, for $m > 1$,

$$\arctan(m) = \frac{\pi}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1) m^{2n+1}}. \tag{44}$$

Substitute (44) into (43), and after tedious calculations, we get

$$\gamma \int_a^y \tanh(cz) e^{-2\gamma z} dz = I_1(y) - I_1(a), \tag{45}$$

where

$$\begin{aligned}
I_1(y) =& \gamma \left(-\frac{\sinh(cy)}{c} + 2\,\frac{\arctan(\mathrm{e}^{cy})}{c} + \frac{\cosh(cy)}{c} \right) e^{cy-2\gamma y} \\
&- \gamma(c - 2\gamma) \Bigg[\frac{(-\cosh(cy) + \sinh(cy))\, \mathrm{e}^{cy-2\,\gamma\, y}}{2\gamma\, c} \\
&+ \frac{2}{c^2} \left(\frac{\pi e^{cy(-\frac{2\gamma}{c}+1)}}{2(-\frac{2\gamma}{c}+1)} - \sum_{n=0}^{\infty} (-1)^n \frac{e^{-(2n+\frac{2\gamma}{c})cy}}{(2n+1)(-2n-\frac{2\gamma}{c})} \right) \Bigg].
\end{aligned}$$

Now substitute (45) into (41) and let $C_1 = I_1(a)$, then

$$\begin{aligned}&\int_a^x 2e^{2\gamma y}\left(\gamma\int_a^y \tanh(cz)e^{-2\gamma z}dz + C_1\right)dy = \int_a^x 2e^{2\gamma y}\gamma I_1(y)dy\\ =&2\gamma\int_a^x\left(-\frac{\sinh{(cy)}}{c} + 2\,\frac{\arctan{(\mathrm{e}^{cy})}}{c} + \frac{\cosh{(cy)}}{c}\right)e^{cy}\\ &-(c-2\gamma)\Bigg[\frac{(-\cosh{(cy)}+\sinh{(cy)})\,\mathrm{e}^{cy}}{2\gamma\, c}\\ &+\frac{2}{c^2}\left(\frac{\pi e^{cy}}{2(-\frac{2\gamma}{c}+1)} - \sum_{n=0}^{\infty}(-1)^n\frac{e^{-2ncy}}{(2n+1)(-2n-\frac{2\gamma}{c})}\right)\Bigg]dy.\end{aligned}$$

The integrals of each of the terms on the right-hand side can be easily calculated. Let $C_2 = 0$, and by summarising the results of calculations, we get the first order solution as

$$\tilde{f}_1(x,\beta) = f_0(x,\beta)(g_1(x,\beta) - g_1(a,\beta)), \tag{46}$$

where

$$\begin{aligned}g_1(x,\beta) =&2\sqrt{2\beta}\left(2\,\frac{\mathrm{e}^{cx}\arctan{(\mathrm{e}^{cx})}}{c^2} - \frac{\ln\left((\mathrm{e}^{cx})^2+1\right)}{c^2} - \frac{\pi e^{cx}}{c^2}\right) + x\\ &-\frac{4\sqrt{2\beta}(c-2\sqrt{2\beta})}{c^2}\left(\frac{cx}{2\sqrt{2\beta}} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-2ncx}}{c2n(2n+1)(-2n-\frac{2\sqrt{2\beta}}{c})}\right).\end{aligned}$$

Note that, (46) is not yet the LT to be inverted. It corresponds to the drift function $h(x) = \tanh(cx)$ while what we want is $h(x) = \theta - \frac{\tanh(cx)}{c}$. But we are almost there. According to the linearity of the perturbed ODE solution (cf. Remark 2.1), the first order solution of $h(x) = -\frac{\tanh(cx)}{c}$ is given by

$$-f_0(x,\beta)\frac{g_1(x,\beta)-g_1(a,\beta)}{c}. \tag{47}$$

On the other hand, by solving the first order solution of a drifted Brownian motion with $h(x) = \theta$, we have

$$\hat{f}_1(x,\beta) = -f_0(x,\beta)\theta(x-a). \tag{48}$$

Using the linearity property again, and combining Eqs. (47), (48), the final first order solution of the hyperbolic OU process in (33) is written as

$$f_1(x,\beta) = f_0(x,\beta)(-\frac{g_1(x,\beta)-g_1(a,\beta)}{c} - \theta(x-a)). \tag{49}$$

Part II (Hyperbolic Bessel Perturbed LT). Similar as in Part I, we first consider the simplified drift function $h(x) = \coth(cx)$ and then generalise the result to $h(x) = c\coth(cx)$. Using the equation in Remark 2.1 again, for the drift function $h(x) = \coth(cx)$, we have

$$\tilde{f}_1 = f_0 \left[\int_a^x 2e^{2\gamma y} \left(\gamma \int_a^y \coth(cz) e^{-2\gamma z} dz + C_1 \right) dy + C_2 \right]. \tag{50}$$

Repeating the same trick as in (42), and applying the integral by parts, we then get

$$\begin{aligned} \gamma \int_a^y \coth(cz) e^{-2\gamma z} dz = \gamma \left(\frac{e^{-cz} + \ln(1-e^{-cz}) - \ln(1+e^{-cz})}{c} \right) e^{cz-2\gamma z} \Big|_{z=a}^{y} \\ - \gamma(c-2\gamma) \int_a^y \left(\frac{e^{-cz} + \ln(1-e^{-cz}) - \ln(1+e^{-cz})}{c} \right) e^{cz-2\gamma z} dz. \end{aligned} \tag{51}$$

For the second term on the right-hand side, we consider the expansions

$$\ln(1-e^{-cz}) = -\sum_{n=1}^{\infty} \frac{e^{-ncz}}{n}, \; \ln(1+e^{-cz}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-ncz}}{n}. \tag{52}$$

Substituting the expansions into (51) and repeating the same calculations as in Part I, we find

$$\gamma \int_a^y \coth(cz) e^{-2\gamma z} dz = I_2(y) - I_2(a) \tag{53}$$

with

$$\begin{aligned} I_2(y) =& \gamma \left(\frac{e^{-cy} + \ln(1-e^{-cy}) - \ln(1+e^{-cy})}{c} \right) e^{cy-2\gamma y} \\ &- \frac{\gamma(c-2\gamma)}{c} \left(-\frac{e^{-2\gamma y}}{2\gamma} + 2\sum_{n=0}^{\infty} \frac{e^{-(2nc+2\gamma)y}}{(2n+1)(2nc+2\gamma)} \right). \end{aligned}$$

The rest calculations follow the same routines as in Part I: substituting (53) into (50), letting $C_1 = I_2(a)$, and continuing the outer-integral calculations; setting $C_2 = 0$ and using again the linearity of the perturbed LT. In the end, for the drift function $h(x) = c\coth(cx)$, the first order solution is given by

$$f_1(x,\beta) = f_0(x,\beta)(g_1(x,\beta) - g_1(a,\beta)), \tag{54}$$

where $g_1(x,\beta)$ is redefined as

$$\begin{aligned} g_1(x,\beta) =& 2\gamma \Bigg(-x - \frac{1-e^{cx}}{c} \ln\left(1-e^{-cx}\right) - \frac{1+e^{cx}}{c} \ln\left(1+e^{-cx}\right) \\ &+ (c-2\gamma) \left(-\frac{x}{2\gamma} + \sum_{n=1}^{\infty} \frac{e^{-2ncx}}{nc(2n+1)(2nc+2\gamma)} \right) \Bigg). \end{aligned}$$

Part III (Explicit Inverse of Perturbed LTs). Recall in (6), the first order perturbed LT is

$$f^{(1)} = f_0 + \epsilon f_1, \tag{55}$$

where depending on the underlying processes, f_1 is given by Eqs. (49) and (54), respectively.

Refer to the g_1-functions in (49) and (54), one can easily find that the LT parameter β involved in the equation above belongs to the following structures (recall that $f_0 = (x-a)/\sqrt{2\pi t^3}e^{-(x-a)^2/2/t}$):

$$e^{-\sqrt{\beta}\sqrt{\alpha}},\ e^{-\sqrt{\beta}\sqrt{\alpha}}\sqrt{\beta},\ e^{-\sqrt{\beta}\sqrt{\alpha}}\frac{\sqrt{\beta}}{\sqrt{\beta}+\xi},\ e^{-\sqrt{\beta}\sqrt{\alpha}}\frac{\beta}{\sqrt{\beta}+\xi},$$

where $\alpha = 2(x-a)^2$ and $\xi = \frac{cn}{\sqrt{2}}$ are constant variables in the inverse transform. Refer to [6, Section 5.6, Eqs. (1), (5), (15), (13)], we have

$$\mathcal{L}^{-1}\left\{e^{-\sqrt{\beta}\sqrt{\alpha}}\right\}(t) = \frac{|x-a|}{\sqrt{2\pi}}t^{-\frac{3}{2}}e^{-\frac{(x-a)^2}{2t}}; \tag{56}$$

$$\mathcal{L}^{-1}\left\{e^{-\sqrt{\beta}\sqrt{\alpha}}\sqrt{\beta}\right\}(t) = \frac{t^{-\frac{5}{2}}}{2\sqrt{\pi}}\left((x-a)^2 - t\right)e^{-\frac{(x-a)^2}{2t}}; \tag{57}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-\sqrt{\beta}\sqrt{\alpha}}\frac{\sqrt{\beta}}{\sqrt{\beta}+\xi}\right\}(t) =& \frac{1}{\sqrt{2\pi}}\left(\frac{|x-a|t^{-\frac{3}{2}}}{2} - cnt^{-\frac{1}{2}}\right)e^{-\frac{(x-a)^2}{2t}} \\ &+ \frac{c^2n^2}{2}e^{cn|x-a|+\frac{c^2n^2t}{2}}\operatorname{Erfc}\left(\frac{|x-a|}{\sqrt{2t}} + \frac{cn}{\sqrt{2}}\sqrt{t}\right);\end{aligned} \tag{58}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-\sqrt{\beta}\sqrt{\alpha}}\frac{\beta}{\sqrt{\beta}+\xi}\right\}(t) =& \frac{t^{-\frac{3}{2}}}{2\sqrt{\pi}}\left(\frac{(x-a)^2}{t} - 1 - cn|x-a| + c^2n^2t\right)e^{-\frac{(x-a)^2}{2t}} \\ &- \frac{c^3n^3}{2\sqrt{2}}e^{cn|x-a|+\frac{c^2n^2t}{2}}\operatorname{Erfc}\left(\frac{|x-a|}{\sqrt{2t}} + \frac{cn}{\sqrt{2}}\sqrt{t}\right).\end{aligned} \tag{59}$$

Finally, substituting Eqs. (56) to (59) into the inverse of (55) with the corresponding f_1-functions in either (49) or (54) and summarising the results, one will find the perturbed density functions in Propositions 3.4 and 3.5. We leave the calculations to the reader and conclude the proof by here. □

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