Monotonous betting strategies in warped casinos *

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Abstract. Suppose that the outcomes of a roulette table are not entirely random, in the sense that there exists a successful betting strategy. Is there a successful 'separable' strategy, in the sense that it does not use the winnings from betting on red in order to bet on black, and vice-versa? We study this question from an algorithmic point of view and observe that every strategy M can be replaced by a separable strategy which is computable from M and successful on any outcome-sequence where M is successful. We then consider the case of mixtures and show: (a) there exists an effective mixture of separable strategies which succeeds on every casino sequence with effective Hausdorff dimension less than 1/2; (b) there exists a casino sequence of effective Hausdorff dimension 1/2 on which no effective mixture of separable strategies succeeds. Finally we extend (b) to a more general class of strategies.

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1 Introduction

A bet in a game of chance is usually determined by two values: the favorable outcome and the wager x one bets on that outcome. If the outcome turns out to be the one chosen, the player gains profit x; otherwise the player loses the wager x. Many gambling systems for repeated betting are based on elaborate choices for the wager x, while leaving the choice of outcome constant. In this work we are interested in such 'monotonous' strategies, which we also call *single-sided*, and their linear combinations (mixtures). Consider the game of roulette, for example, and the binary outcome of red/black.[1] Perhaps the most infamous roulette system is the *martingale*,[2] where one constantly bets on a fixed color, say *red*, starts with an initial wager x and doubles the wager after each loss. At the first winning stage all losses are then recovered and an additional profit x is achieved. Such systems rely on the fairness of the game, in the form of a law of large numbers that has to obeyed in the limit (and, of course, require unbounded initial resources in order to guarantee success with probability 1). In the example of the martingale the relevant law is that, with probability 1, there must be a round where the outcome is red. Many other systems have been developed that use more tame series of wagers (compared to the exponential increase of the martingale), and which appeal to various forms of the law of large numbers.[3]

When the casino is biased, i.e. the outcomes are not entirely random, we ought to be able to produce more successful strategies. Suppose that we bet on repeated coin-tosses, and that we are given the information that the coin has a bias. In this case it is well known that we can define an effective strategy that, independent of the bias of the coin (i.e. which side the coin is biased on, or even any lower bounds on the bias), is guaranteed to gain unbounded capital, starting from any non-zero initial capital. This strategy, as we explain in §2.3, is the mixture of two single-sided strategies, where the first one always bets on heads and the second one always bets on tails. A slightly modified strategy is successful on every coin-toss sequence X except for the case that the limit of the relative frequency of heads exists and is 1/2. The same kind of strategy exists for the case where the relative frequency of heads is 1/2, but beyond some point the number of tails is never smaller than the number of heads (or vice-versa). These examples show that many typical betting strategies are *separable* in the sense that they can be expressed as a the sum of two single-sided strategies. In the following we refer to any binary sequence which is produced by a (potentially partially) random process, as a *casino sequence*. Note that if a separable strategy succeeds along a casino sequence, one of its single-sided parts has to succeed. The only case where separability is stronger than single-sidedness is when we consider success with respect to classes of casino sequences.

A casino sequence may have a (more subtle) bias while satisfying several known laws of large numbers, such as the relative frequency of 0s tending to 1/2. Formally, we can say that a casino sequence X is biased if there is an 'effective' (as in 'constructive' or 'definable') betting strategy which succeeds on X, i.e. produces an unbounded capital, starting from a finite initial capital. By adopting stronger or weaker formalizations of the term 'effective' one obtains different strengths of bias, or as we usually say, *non-randomness* of X. In general, 'effective' means that the strategy is definable in a simple way, such as being programmable in a Turing machine. Suppose that we know that the casino sequence X has a bias in this more general sense, i.e. there exists some 'effective' betting strategy which succeeds on it. The starting point of the present

[1] Roulettes have a third outcome 0, which is neither red nor black, and which gives a slight advantage to the house. For simplicity in our discussion we ignore this additional outcome.

[2] for the origin of this term, its use as a betting system and its adoption in mathematics, see Mansuy [2005] and Snell [1982].

[3] Well-known systems of this kind are: the D'Alembert System, the Fibonacci system, the Labouchère system or split martingale, and many others. See, for example, `https://www.roulettesystems.com`.

article is the following question:

Is it possible to succeed on any such warped casino sequence with a single-sided 'effective' betting strategy, i.e. one that can only place bets on 0 or only on 1? (1)

In other words, can any 'effective' betting strategy be replaced by a single-sided 'effective' betting strategy without sacrificing success? An equivalent way to ask this question is as follows.

Suppose that we are betting with the restriction that we cannot use our earnings from the successful bets on 0s in order to bet on 1s, and vice-versa. Can we win on any casino-sequence X which is 'biased' in the sense that there is an (unrestricted) strategy which wins on X? (2)

We will see that, depending on the way we formalise the term 'effective', and especially the term *effective monotonous betting* these questions can have a positive or negative (or even unknown) answer.

Our results. A straightforward interpretation of 'effective' is computable, in the sense that there is a Turing machine that decides, given each initial segment of the casino sequence:

(a) how much of the current capital to bet; (b) which outcome to bet on.

These choices, in combination with the revelation of the outcome, determine the capital at the beginning of the next betting stage. In §3 we show that in this case questions (1) and (2) have a positive answer. Another formalisation of 'effective' which is very standard in computability and algorithmic information theory (and used in the standard definition of algorithmic randomness) is 'computably enumerable'. When applied to betting strategies this gives a notion which is equivalent to infinite mixtures of strategies which are generated by a single Turing machine, see the introductory part of §2. There are two very different ways that one can define computably enumerable monotonous strategies:

(i) *Uniform way:* as the mixture (linear combination) of a computable *family of monotonous strategies* with bounded total initial capital;

(ii) *Non-uniform way:* as a monotonous strategy that can be expressed as the mixture of a computable family of strategies with bounded total initial capital.

In the uniform case we show that questions (1) and (2) have negative answers. In fact, we show that there are casino sequences X on which mixtures of computable families of strategies generate infinite capital exponentially fast, in the sense that[4]

$$\limsup_n \frac{M(X\upharpoonright_n)}{\alpha^n} = \infty \quad \text{where } \alpha \in (1, \sqrt{2}) \text{ and } M \text{ is the capital after the first } n \text{ bets on } X, \tag{3}$$

where $X\upharpoonright_n$ denotes the first n bits of X, but no strategy under (i) succeeds. We also show the converse, i.e. that if a computably enumerable strategy (i.e. a mixture of computable family of strategies) M exists such that $\limsup_n M(X\upharpoonright_n)/\alpha^n = \infty$ for some $\alpha > \sqrt{2}$, then there exists a single-sided computably enumerable strategy N which succeeds on X, in the sense that $\lim_n N(X\upharpoonright_n) = \infty$. We will see that these results can also be stated in terms of the effective Hausdorff dimension of the casino sequence. Under the uniform case we also consider a more general class of strategies, which we call *decidably-sided*, and which are not necessarily monotonous, but there is a computable prediction (or choice) function which indicates the favorable outcome at each state. We then generalise our previous arguments and show that there is a casino

[4] .

sequence and a computably enumerable betting strategy M that strongly succeeds on it as before, in the sense of (3), but such that no decidably-sided computably enumerable strategy succeeds on it.

Monotonous strategies under the non-uniform clase (ii) are intuitively more powerful, as we explain in §2.2, and our arguments do not appear to be adequate for answering questions (1) and (2) in this case. The study of the power of strategies in (ii) is quite interesting from the point of view of stochastic processes, as it relates to key concepts such as martingale decompositions, variation and various forms of boundedness or integrability. Questions (1) and (2) under (ii) are also directly relevant to a question about the separation of two randomness notions in algorithmic information theory, asked by Kastermans (see [Downey, 2012] and [Downey and Hirschfeldt, 2010, §7.9]). As we point out in §5, a positive answer of (1) or (2) for the case of strategies under (ii) would give a very simple and elegant positive answer to Kasterman's question.

Outline of the presentation. The concept of a betting strategy in terms of martingale functions is formalised in the first part of §2. Monotonous strategies are formalised in §2.1 and effective versions of mixtures of monotonous strategies are given in §2.2, along with relevant characterizations in terms of computable enumerability. In §2.3 we show that many types of betting are monotonous and in §2.4, after recalling that Hausdorff dimension is expressible in terms of speed of martingale success, we use these facts in order to show that there exists a separable strategy which succeeds in all casino sequences of effective Hausdorff dimension $< 1/2$. In §3 we first describe a decomposition of computable martingales into two single-sided (orthogonal) martingales, which provides the positive answer to questions (1) and (2) stated in the introductory discussion, for the case of computable strategies. We then give a detailed argument establishing a strong negative answer of the same questions for the special case of a single separable strategy. This argument is then used in a modular way in §4 in order to obtain a proof of the full result, with respect to every possible strategy that is expressible as a mixture of a computable family of separable martingales. Finally in §4.4 we generalize this result to the more general class of decidably-sided strategies. Concluding remarks and a critical discussion of our results, along with open problems and directions for future investigations are given in §5.

2 Monotonous betting strategies and their mixtures

Betting strategies are formalized by martingales[5] which are used in order to express the capital after each betting stage and each casino outcome. Formally, a *martingale* in the space of binary outcomes is a function $M : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ from binary strings to the non-negative real numbers, with the property that for all $\sigma \in 2^{<\omega}$:

$$2 \cdot M(\sigma) = M(\sigma * 0) + M(\sigma * 1). \tag{4}$$

If the equality is replaced with '$\geq$' then M is called a *supermartingale*.[6] Probabilistically, such a function M can be seen as a martingale stochastic process (Y_s) *relative* to the underlying fair coin-tossing stochastic process (I_s), where I_s is the outcome of the sth coin-toss which can be 0 or 1 with equal probability 1/2, so that:

(a) Y_s is measurable in (i.e. determined by the outcome of) $I_i, i \leq s$;

[5] This is a mathematical notion and different than the martingale betting system that we discussed in §1. In mathematics, martingales were introduced by Lévy [1937] and extended by Ville [1939] who also gave them this name. See Doob [1971] for a classic and brief exposition of martingales in probability.

[6] Supermartingales can be viewed as 'leaky martingales' which may potentially lose some capital at each betting position.

(b) by (4) the expectation of Y_{s+1} given $I_i, i \leq s$ equals Y_s.

The definition of a martingale in (4) as a deterministic function relates to its probabilistic interpretation in the same way that a random variable can be seen as a deterministic function from a probability space to $\mathbb{R}$. If we view martingales M as deterministic functions satisfying (4), and if we *require them to be non-negative*, then they provide a formalisation of a betting strategy on an infinite coin-tossing game, where $M(\sigma)$ denotes the capital at position σ. Non-negativity expresses the requirement that the player cannot borrow money after a bankruptcy, i.e. upon the loss of all the capital, the game ends. Informally a bet consists of the favorable outcome (0 or 1) and the *wager*, which is the amount that will be won or lost after the outcome is revealed. For convenience, we combine both of these parameters into the definition of the wager, whose sign reveals the favorable outcome:

$$w_M(\sigma) := M(\sigma * 1) - M(\sigma) \quad \text{ is the } \textit{wager at state } \sigma. \tag{5}$$

Hence if $w_M(\sigma) > 0$ then the favorable outcome in this bet is 1; if $w_M(\sigma) < 0$ then the favorable outcome is 0. If $w_M(\sigma) = 0$ then no bet is placed at position σ. Wagers are usually called *martingale differences* in probability texts. We say that M *succeeds* on X if

$$\limsup_n M(X \restriction_n) = \infty. \tag{6}$$

In order to consider realistic strategies it is natural to require that the martingales are definable or have some effectivity properties, for example that they are *computable* or *enumerable* by a Turing machine.

Definition 2.1 (Computably enumerability of martingales)**.** A martingale $M : 2^{<\omega} \to \mathbb{R}^+$ is called *l.c.e.* if $M(\sigma)$ can be approximated by an increasing computable sequence of rationals, uniformly in σ. Moreover we say that M is *strongly l.c.e.* if is it left-c.e. and the wagers $w_M(\sigma)$ can be approximated by strictly monotone computable sequence of rationals, uniformly in σ.[7]

Computable and left-c.e. martingales can be used as a foundation of algorithmic information theory, see [Downey and Hirschfeldt, 2010, §13.2], [Li and Vitányi, 1997] or [Bienvenu et al., 2009]. A binary sequence to be algorithmically random if no left-c.e. martingale M *succeeds* on it in the sense of (6).

Martingales and algorithmic randomness. It turns out that any left-c.e. martingale M can be transformed into a left-c.e. martingale N such that $\lim_n N(X \restriction_n) = \infty$ for each X such that (6) holds. The betting strategies (or *unpredictability*) approach to algorithmic randomness is equivalent to the other two traditional approaches, namely the *incompressibility* approach (through Kolmogorov complexity) and the *measure-theoretic* approach (through statistical tests). So a real X is Martin-Löf random (i.e. roughly speaking, avoids all effective null sets) if and only if there exists some constant c for which $\forall n\ K(X \restriction_n) > n - c$, where K denotes the prefix-free Kolmogorov complexity of X, if and only if no left-c.e. (super)martingale succeeds on X. The equivalence of the martingale approach with the other two, established in Schnorr [1971a], is based on the *Kolmogorov inequality* (sometimes known as Ville's inequality as it appears in Ville [1939]) which will be used in §3, §4 and says that if M is a martingale then:

$$\sum_{\sigma \in S} 2^{-|\sigma|} \cdot M(\sigma) \leq M(\lambda) \quad \text{ for each prefix-free set of strings } S \tag{7}$$

where λ denotes the empty string. If S covers the whole space then equality holds, giving a version of the familiar fairness condition described by the martingale property.

[7]The reader may verify the following redundancy in the second clause of the definition: if the wagers $w_M(\sigma)$ can be approximated by an increasing computable sequence of rationals, uniformly in σ and the initial capital $M(\lambda)$, where λ is the empty string, is left-c.e. (i.e. has a computable increasing rational approximation) then necessarily M is a left-c.e. martingale.

2.1 Monotonous strategies as martingales

We formally define strategies that bet in a monotonous fashion, in terms of martingales.

Definition 2.2 (Single-sided strategies)**.** A martingale M is 0-sided if $M(\sigma * 0) \geq M(\sigma * 1)$ for all σ; it is 1-sided if $M(\sigma * 1) \geq M(\sigma * 0)$ for all σ. We say that M is single-sided if it is either 0-sided or 1-sided. We say that M is strictly single-sided if it is single sided and $M(\sigma * 0) \neq M(\sigma * 1)$ for all σ.

A *prediction function* f is a function from $2^{<\omega}$ to $\{0, 1\}$. We say that $i < |\sigma|$ is a correct f-guess with respect to σ if $f(\sigma \upharpoonright_i) = \sigma(i)$; otherwise we say that i is a false f-guess with respect to σ. According to the two components (a), (b) of a betting strategy discussed in §1, a prediction function can be seen as (b).

Definition 2.3 (Decidably-sided strategies)**.** Given a prediction function f, a martingale M is f-sided if any bias on the outcomes is decided by f; formally, if for all σ, i if $M(\sigma * i) > M(\sigma)$ then $f(\sigma) = i$, and similarly if $M(\sigma * i) < M(\sigma)$ then $f(\sigma) = 1 - i$. A martingale M is decidably-sided if its favorable outcome is decidable, in the sense that it is f-sided for a (total) computable prediction function f.

Decidably-sided strategies can be seen as single-sided betting strategies modulo some effective re-naming of 0s and 1s. Another restricted strategy that we discussed informally in (2) is when the bets on 0s and the bets on 1s are based on separated capital pools, with any winnings being returned to them, and losses taken from them, in a disjoint fashion. These strategies are modeled by *separable martingales* which are martingales that can be written as the sum of a 0-sided and a 1-sided martingale.

Facts and non-facts about monotonous betting. It is clear that f-sided and separable martingales are closed under (countable, subject to convergence of initial capitals) addition and multiplication by a constant. Many of the facts about left-c.e. martingales in the beginning of §2 also hold for the restricted martingales introduced above, by similar proofs. Assuming that f is computable:

> if M is an f-sided martingale then there is an f-sided martingale N with $\lim_n N(X \upharpoonright_n) = \infty$ for all X on which M succeeds, in the sense of (6). The same holds even if we replace 'f-sided' with 'separable' or 'decidably-sided', or qualify M, N as left-c.e. or computable.

The proof is a simple adaptation of the standard argument, the so-called *savings trick*, (see [Downey and Hirschfeldt, 2010, Proposition 6.3.8]). Since algorithmic randomness can be defined with respect to a class of effective (super)martingales, each of the restricted martingale notions that we have discussed, left-c.e. or computable, corresponds to a randomness notion. Separating these notions is often a matter of adapting existing methods on this topic, such as [Nies, 2009, Chapter 7].

Theorem 2.4 (Partial computable strategies vs single-sided left-c.e. strategies)**.** *There exists X such that a 0-sided left-c.e. martingale succeeds on X and no partial computable (super)martingale succeeds on X.*

The proof of Theorem 2.4 is a straightforward adaptation of the arguments in [Nies, 2009, §7.4] and is thus left to the reader as an exercise. Our results in §3 and §4 can also be viewed as separations of randomness notions, but their proofs require a novel argument. On the other hand, certain caution is needed as some basic facts about (super)martingales and their effective versions, no-longer hold in the presence of monotonousness. It is crucial to observe that *the difference of two single-sided martingales is not always single-sided*, even if it is positive and even if they both favor the same outcome. This is the reason why the two notions (i),(ii) of computably enumerable monotonous strategies discussed in §1 are quite different. Another issue is that under monotonousness, supermartingales are not interchangeable with martingales. Classically, every supermartingale is bounded above by a martingale, and this is also true for computable and left-c.e. supermartingales (the left-c.e. case is not straightforward; see [Downey and Hirschfeldt, 2010, §6.3]). Although this fact is also true for single-sided supermartingales in the non-effective and computable cases, it can be shown to fail for left-c.e. single-sided supermartingales.

2.2 Mixtures of monotonous strategies, enumerability and approximations

By the *mixture* of a finite or countable family (M_i) of non-negative martingales we mean the sum $M = \sum_i M_i$. In this terminology, there are two implicit assumptions: (a) the sum is bounded, in the sense that the total initial capital of the M_i is finite: $\sum_i M_i(\lambda) < \infty$; (b) since we typically deal with effective or constructive strategies, we assume that (M_i) has the same complexity, for example it is uniformly computable. Mixtures of computable families of martingales allow for more powerful betting strategies since, although (M_i) is uniformly computable, the values of the capital $M = \sum_i M_i$ can only be approximated by a computable increasing sequence, uniformly in the argument. Martingales M with the latter approximation property are left-c.e. according to Definition 2.1 and are conceptually interesting since, although the current capital $M(\sigma)$ and wager $w_M(\sigma)$ are measurable, i.e. determined, from the state σ, a constructive (computable) observer only has access to a approximations of them. Hence even the favorable outcome may not be computable, while for strongly left-c.e. martingales a computable observer has access to the favorable outcome as well as a lower bound converging to the absolute value of the current wager.

Mixtures, enumerable strategies and optimality. The mixture of a computable family of martingales is a left-c.e. martingale. Moreover the mixture of a computable family of f-sided martingales is an f-sided strongly left-c.e. martingale. In the following, we point out that the converse of these facts is true: every left-c.e. martingale can be written as the mixture of a computable family of martingales; similarly, every strongly left-c.e. strictly f-sided left-c.e. martingale can be written as the mixture of a computable family of strictly f-sided martingales. These facts provide useful approximations for left-c.e. monotonous martingales, which will be used in §3, §4. The reason that such respresentations are needed in the proofs that involve diagonalization, is the somewhat surprising lack of universality in the class of left-c.e. martingales. By [Downey et al., 2004] there exists no effective enumeration of all left-c.e. martingales. This is usually an inconvenience in arguments which involve diagonalisation against all left-c.e. martingales, and a reason why it is often convenient to work with supermartingales (recall the discussion in §2.1 that effective martingales and supermartingales are exchangeable). Since there exists a uniform enumeration of all left-c.e. supermartingales, there exists a left-c.e. supermartingale M which is *optimal*, in the sense that any other left-c.e. supermartingale is $\mathbf{O}(M)$, i.e. multiplicatively dominated by M. On the other hand, by [Downey et al., 2004] there is no optimal left-c.e. martingale M, i.e. such that any other left-c.e. martingale is $\mathbf{O}(M)$. Unfortunately, *our arguments are specific to martingales* and do not apply to supermartingales. This, along with the fact discussed in so we need to deal with the fact that, as discussed in §2.1, supermartingales are not exchangeable with martingales under monotonousness, means that we cannot use universality in our arguments.

Lemma 2.5 (Left-c.e. martingales as effective mixtures)**.** *A martingale is left-c.e. if and only if it can be written as the sum of a uniformly computable sequence of martingales.*

Proof. If (N_i) is a uniformly computable sequence of martingales and $\sum_i N_i(\lambda) < \infty$ then it is well-known that $\sigma \mapsto \sum_i N_i(\sigma)$ is a left-c.e. martingale. For the converse, assume that M is a left-c.e. martingale and let (M_s) be a left-c.e. approximation to it so that $M_{s+1}(\sigma) > M_s(\sigma)$ for all s, σ. We define a family (N_i) of martingales as follows. Inductively assume that $N_i, i < k$ have been defined, they are martingales, and

$$S_k(\sigma) < M(\sigma) \quad \text{for all } \sigma, \text{ where } S_k := \sum_{i<k} N_i. \tag{8}$$

Consider a stage s_0 such that $M_{s_0}(\lambda) > \sum_{i<k} N_i(\lambda)$ and let $N_k(\lambda) = M_{s_0}(\lambda) - S_k(\lambda)$. Then for each σ suppose inductively that we have defined $N_k(\sigma)$ in such a way that $N_k(\sigma) + S_k(\sigma) \le M_t(\sigma)$ for some stage t. Since M is a martingale, this means that there exists some larger stage s such that:

$$M_s(\sigma * 0) + M_s(\sigma * 1) \ge 2N_k(\sigma) + 2S_k(\sigma) = 2N_k(\sigma) + (S_k(\sigma * 0) + S_k(\sigma * 1)). \tag{9}$$

Then we let $N_k(\sigma * i), i = \{0, 1\}$ be two non-negative rationals such that:

(a) $N_k(\sigma * 0) + N_k(\sigma * 1) = 2N_k(\sigma)$;

(b) $N_k(\sigma * i) + S_k(\sigma * i) \le M_s(\sigma * i)$ for each $i = \{0, 1\}$.

This concludes the inductive definition of N_k and also verifies the property (8) for $k + 1$ in place of k. Note that the totality of each N_i is guaranteed by the fact that M is a martingale. It remains to show that

$$\lim_k S_k(\sigma) = M(\sigma) \quad \text{for each } \sigma. \tag{10}$$

By the definition of $N_i(\lambda)$, it follows that (10) holds for $\sigma = \lambda$. Assuming (10) for σ, we show that it holds for $\sigma * i, i \in \{0, 1\}$. We have

$$M(\sigma * 0) + M(\sigma * 1) - S_k(\sigma * 0) - S_k(\sigma * 1) = 2M(\sigma) - 2S_k(\sigma) = 2(M(\sigma) - S_k(\sigma)), \tag{11}$$

so by (10) we have: $\lim_k S_k(\sigma * 0) + \lim_k S_k(\sigma * 1) = M(\sigma * 0) + M(\sigma * 1)$. By (8) applied to $\sigma * 0$ and $\sigma * 1$ we get $\lim_k S_k(\sigma * i) = M(\sigma * i)$ for $i \in \{0, 1\}$, as required. This concludes the inductive proof of (10). □

Mixtures, monotonous betting and intermediate bets. Recall the two ways (i), (ii) that monotonous betting can be considered for mixtures of strategies. We will show that for mixtures of computable families of monotonous strategies, these two formulations are essentially equivalent to the two notions of computable enumerability of martingales in Definition 2.1. The difference between (i) and (ii) is clear if we view a mixture S at a state σ as an infinite countable stack of bets that are being placed on the initial segments of σ. The crucial property of effective single-sided strategies S under (i), is that they are effectively approximated by single-sided strategies (S_i) such that for each $n < m$, *the intermediate bets $S_m - S_n$ are also single-sided*. Since in general the difference of single-sided strategies may not be single-sided, this property may not be present under clause (ii). A computable observer can only access a certain approximation to S at each stage, i.e. a certain finite initial segment of the bets that compose S. At later stages the observer has access a more accurate approximation: *the intermediate bets express the error of the first observation with respect to the current one*. For an analogue of Lemma 2.5 in the case of monotonous left-c.e. martingales (non-uniform case (ii)) we require strict monotonousness in the sense of Definitions 2.2 and 2.3, i.e. that a non-empty bet is placed at every state. This requirement is not essential, as Lemma 2.6 shows.

Lemma 2.6. *If f is a computable prediction function, then for each left-c.e. f-sided martingale M we can effectively obtain a left-c.e. strictly f-sided martingale $\hat{M}$ such that for each X with $\limsup_s M(X \upharpoonright_s) = \infty$ we have $\limsup_s \hat{M}(X \upharpoonright_s) = \infty$. Hence if no strictly f-sided left-c.e. martingale succeeds on a real Y, then no f-sided left-c.e. martingale succeeds on Y.*

Proof. Let N be the computable martingale which starts with $N(\lambda) = 1$ and at each σ, it bets half of $N(\sigma)$ on $f(\sigma)$. Define $\hat{M} = M + N$ so $\hat{M}$ is clearly f-sided and succeeds on every real that M does. Since $N(\sigma) > 0$ for all σ it follows that N is strictly f-sided. Then $\hat{M}(\sigma * f(\sigma)) - \hat{M}(\sigma * (1 - f(\sigma)))$ equals

$$(M(\sigma * f(\sigma)) - M(\sigma * (1 - f(\sigma))) + (N(\sigma * f(\sigma)) - N(\sigma * (1 - f(\sigma)))$$

which is larger than 0 as required, since M is f-sided and N is strictly f-sided. □

Lemma 2.7 (Monotonous left-c.e. martingales as mixtures)**.** *For every computable prediction function f and every left-c.e. strictly f-sided martingale M, there exists a uniformly computable sequence (N_i) of martingales such that the partial sums $S_n = \sum_{i<n} N_i$ are f-sided and converge to M.*

Proof. The proof is a simple adaptation of the proof of Lemma 2.5, so we may refer to the displayed equations in that proof, although the parameters have a modified meaning that we determine below. We give the proof of the case of single-sided martingales, as the case of decidably-sided martingales is entirely analogous. Without loss of generality, assume that M is a left-c.e. and 0-sided martingale. and let (M_s) be a left-c.e. approximation to it so that $M_{s+1}(\sigma) > M_s(\sigma)$ for all s, σ. We define a computable family (N_i) of martingales: inductively assume that $N_i, i < k$ have been defined and are martingales, and

$$S_k := \sum_{i<k} N_i \quad \text{is 0-sided, and for all } \sigma, \quad S_k(\sigma) < M(\sigma) \tag{12}$$

Consider a stage s_0 such that $M_{s_0}(\lambda) > \sum_{i<k} N_i(\lambda)$ and let $N_k(\lambda) = M_{s_0}(\lambda) - S_k(\lambda)$. Given σ, suppose inductively that we have defined $N_k(\sigma)$ in such a way that $N_k(\sigma) + S_k(\sigma) \leq M_t(\sigma)$ for some stage t, and for each $\rho \prec \sigma$ we have $S_{k+1}(\rho * 0) \geq S_{k+1}(\rho * 1)$. Since M is a 0-sided martingale, there exists some $s > t$ such that (9) and $M_s(\sigma * 0) > M_s(\sigma * 1)$. Then we let $N_k(\sigma * i), i = \{0, 1\}$ be two non-negative rationals such that:

(a) $N_k(\sigma * 0) + N_k(\sigma * 1) = 2N_k(\sigma)$;

(b) $N_k(\sigma * i) + S_k(\sigma * i) \leq M_s(\sigma * i)$ for each $i = \{0, 1\}$.

(c) $N_k(\sigma * 1) + S_k(\sigma * 1) \leq N_k(\sigma * 0) + S_k(\sigma * 0)$ for each $i = \{0, 1\}$.

This concludes the inductive definition of N_k and also verifies the property (12) for $k + 1$ in place of k.

> *Remark on the definition:* If $M_s(\sigma * 0) - M_s(\sigma * 1) < M_t(\sigma * 0) - M_t(\sigma * 1)$, it is possible that the chosen values satisfy $N_k(\sigma * 1) > N_k(\sigma * 0)$, in which case N_k is not 0-sided. (13)

The totality of each N_i is guaranteed by the fact that M is a strictly 0-sided martingale. It remains to show (10). From the definition of each $N_i(\lambda)$, it follows that (10) holds for $\sigma = \lambda$. Assuming (10) for σ, we show that it holds for $\sigma * i, i \in \{0, 1\}$. We have (11) as before, so by (10) we have: $\lim_k S_k(\sigma * 0) + \lim_k S_k(\sigma * 1) = M(\sigma * 0) + M(\sigma * 1)$. By (12) applied to $\sigma * 0$ and $\sigma * 1$ we get $\lim_k S_k(\sigma * i) = M(\sigma * i)$ for $i \in \{0, 1\}$, as required. This concludes the induction for (10). □

Lemma 2.8 (Monotonous strongly left-c.e. martingales as mixtures)**.** *For every computable prediction function f and every strictly f-sided strongly left-c.e. martingale M, there exists a uniformly computable sequence (N_i) of f-sided martingales such that the partial sums $S_n = \sum_{i<n} N_i$ converge to M.*

Proof. We do the proof for the case when M is strictly 0-sided, as the more general case is entirely similar. By (13) the application of the construction in Lemma 2.7 to the given M does not ensure that the N_i are 0-sided. In order to achieve this, we note that since M is assumed to be strongly left-c.e. and 0-sided, there exists a computable left-c.e. approximation (M_s) to M such that $M_t(\sigma) > M_s(\sigma)$ and $M_s(\sigma * 0) - M_s(\sigma * 1) < M_t(\sigma * 0) - M_t(\sigma * 1)$ for all $s < t$ and all σ. Using this approximation (M_s) we can apply the construction in Lemma 2.7 with the extra clause (d): $N_k(\sigma * 1) \le N_k(\sigma * 0)$ in the induction step for the definition of $N_k(\sigma * i), i = \{0, 1\}$. This extra clause does not affect the existing argument, hence the constructed (N_k) is a computable family of martingales such that their mixture converges to M. In addition, clause (d) in the inductive definition directly guarantees that each N_k are 0-sided. □

The constructions in §3, §4 rely on the existence of certain 'canonical' effective approximations.

Definition 2.9 (Canonical approximations of monotonous martingales)**.** Given a computable prediction function f and a left-c.e. f-sided martingale N, a *canonical approximation* to N is a computable family (S_i) of f-sided martingales that converge to N such that each $S_{i+1} - S_i$ is also an f-sided martingale.

By Lemma 2.7 every strictly decidably-sided strongly left-c.e. martingale has a canonical approximation. Given single-sided martingales N, T, we say that (M_s) is a canonical approximation to the separable martingale $M = N + T$ if $M_s = N_s + T_s$ for canonical approximations $(N_s), (T_s)$ of N, T respectively.

2.3 Monotonous betting on a biased coin

We give two examples of types of biases can be exploited through single-sided or separable strategies, establishing basic properties of monotonous betting that will be used mainly in §2.4.

Monotonous betting on Villes' casino sequence. A well-known[8] debate in the early days of probability occurred between the competing approaches of Kolmogorov, which won the debate, and the frequentist-based approach of von Mises, for the establishment of the foundations of probability. A significant factor for the loss of support to von Mises' theory was a certain casino sequence constructed by Ville [1939][9] which is 'random' with respect to any given countable collection of choice sequences (a basic tool in von Mises' strictly frequentist approach) but is biased according to a well-accepted statistical test: although the frequency of 0s approaches 1/2, in all initial segments this frequency never drops below 1/2. We point out that the bias in Villes' well-known example is exploitable by computable monotonous betting. In order to see this, let z_n, o_n be the number of 0s and 1s respectively, in the first n bits of Ville's casino sequence, so that $z_n \ge o_n$ for all n. In the case where $\sup_n(z_n - o_n) = \infty$ our strategy is to start with capital 1, and bet wager 1 on outcome 0 at each step. In the case where $\limsup_n(z_n - o_n) := k < \infty$, given k and a stage t such that for all $n \ge t$ we have $z_n - o_n \le k$, we can used the following strategy: given any stage $s_0 > t$, find some $n \ge s_0$ such that $z_n - o_n = k$ and at this n bet on 1. In order to avoid the dependence of this strategy on the parameters k, t, we can consider a mixture including a strategy for each possible pair (k, t), with initial capital for the s-th strategy equal to 2^{-s} (so that the total initial capital is finite).Note that in the first case the strategy is 0-sided and in the second case it is 1-sided; moreover in both cases, under the respective assumption, the strategies are successful on Ville's casino sequence. The mixture of these two strategies is a computable separable strategy and is successful on Ville's sequence.

[8]Short expositions of the debate in relation to the notion of algorithmic randomness can be found on textbooks on this topic such as [Li and Vitányi, 1997, §1.9] and [Downey and Hirschfeldt, 2010, §6.2]. Extended discussions of the philosophical underpinnings of this debate can be found in van Lambalgen [1987] and the more recent Blando [2015].

[9]An English translation can be found at `http://www.probabilityandfinance.com/misc/ville1939.pdf`. Simpler proofs of Ville's theorem appear in Lieb et al. [2006] and [Downey and Hirschfeldt, 2010, §6.5]

Monotonous betting for skewed or non-existent limiting frequency. Given a casino sequence X with limiting frequency of 0s different than 1/2, there is a single-sided betting strategy that is successful on X. Moreover there is a separable martingale which succeeds on every such X, irrespective of whether the frequency is above or below 1/2, or even how much it differs from 1/2. A slightly more general version of these facts, is the following form of Hoeffding's inequality which we prove via of betting strategies, and which will be used in our later arguments..

Lemma 2.10 (Hoeffding for prediction functions)**.** *Given $q > 1/2$, $n \in \omega$ and a prediction function f, the number of strings in 2^n for which the number of correct f-guesses is more than qn is at most $r_q^{-n} \cdot 2^n$, where $r_q > 1$ is a function of q. So the number of strings in 2^n for which the number of correct f-guesses is in the interval $((1-q)n, qn)$ is at least $2^n \cdot (1 - 2r_q^{-n})$.*

Proof. Given f, let z_σ denote the number of correct f-guesses with respect to σ, and let o_σ be the number of false f-guesses with respect to σ. For each $q > 1/2$, consider the function $d : 2^{<\omega} \to \mathbb{R}^+$ defined by $d(\lambda) = 1$ and $d(\sigma) = 2^{|\sigma|} \cdot q^{z_\sigma} \cdot (1-q)^{o_\sigma}$. Note that, if $f(\sigma) = 0$ then:

$$d(\sigma * 0) + d(\sigma * 1) = 2^{|\sigma|+1} \cdot \left(q^{z_\sigma+1} \cdot (1-q)^{o_\sigma} + q^{z_\sigma} \cdot (1-q)^{o_\sigma+1}\right) = 2^{|\sigma|+1} \cdot q^{z_\sigma} \cdot (1-q)^{o_\sigma} = 2d(\sigma).$$

The same is true in the case that $f(\sigma) = 1$, so that d is a martingale, which bets $|d(\sigma*0) - d(\sigma)| = (2q-1)d(\sigma)$ on the prediction of f at σ. For each σ let $p_\sigma = z_\sigma/|\sigma|$, so that $1 - p_\sigma = o_\sigma/|\sigma|$. Suppose that $p_\sigma > q$. Then

$$d(\sigma) = \left(2 \cdot q^{p_\sigma} \cdot (1-q)^{1-p_\sigma}\right)^{|\sigma|} > \left(2 \cdot q^q \cdot (1-q)^{1-q}\right)^{|\sigma|}$$

where the second inequality holds because the function $x \mapsto 2q^x(1-q)^{1-x}$ is increasing[10] in $(0,1)$ when $q > 1/2$. Again by considering the derivatives, we can see that the function $q \mapsto q^q \cdot (1-q)^{1-q}$ is decreasing in $(0, 1/2)$, increasing in $(1/2, 1)$ and it has a global minimum in $(0,1)$ at $q = 1/2$, at which point it takes the value 1/2. So if we let $r_q := 2q^q \cdot (1-q)^{1-q}$ and recall that $q > 1/2$ we get $r_q > 1$ and $d(\sigma) \geq r_q^{|\sigma|}$ for each σ with $p_\sigma > q$. From Kolmogorov's inequality in then follows that, if t_n is the number of strings $\sigma \in 2^n$ with $p_\sigma > q$, then $t_n \cdot 2^{-n} < r_q^{-n}$. So $t_n < r_q^{-n} \cdot 2^n$ as required. □

Computable single-sided randomness and frequency. Lemma 2.10 says that for each total prediction function f, with high probability the number of correct f-guesses along a binary string σ are concentrated around $|\sigma|/2$. In fact, there exists a separable computable martingale which succeeds on every stream X with the property that the proportion of correct f-guesses along X does not reach limit 1/2. For each $q \in (1/2, 1)$ let $T_q(\lambda) = 1$, and define $T_q(\sigma) = 2^{|\sigma|} \cdot q^{z_\sigma} \cdot (1-q)^{o_\sigma}$ where z_σ is the number of correct f-guesses with respect to σ and o_σ is the number of false f-guesses with respect to σ. By the proof of Lemma 2.10, $T_q(\sigma)$ is a martingale and $\limsup_s T_q(X \upharpoonright_n) = \infty$ for each X such that $\limsup_s z_{X\upharpoonright_n}/n > q$. Similarly, $T_q(\sigma)$ is a martingale for each $q < 1/2$, and $\limsup_s T_q(X \upharpoonright_n) = \infty$ for each X such that $\limsup_s z_{X\upharpoonright_n}/n < q$. Let $q_i = 1/2 + 2^{-i-1}$ and $p_i = 1/2 - 2^{-i-1}$ for each i and define:

$$N(\sigma) = \sum_i 2^{-i} \cdot T_{q_i}(\sigma) + \sum_i 2^{-i} \cdot T_{p_i}(\sigma).$$

Then N is a computable martingale and by the properties of T_{q_i}, T_{p_i}, it succeeds on every X for which the proportion of correct f-guesses does not tend to 1/2. In the case that f is the constant zero function T_q is

[10] The derivative of $x \mapsto q^x(1-q)^{1-x}$ is $(\log q - \log(1-q)) \cdot (1-q)^{1-x} \cdot q^x$.

0-sided, which implies the following fact, where 'computably single-sided random' is a sequence where no computable single-sided (super)martingale succeeds.

> There exist computable families (N_i), (T_i) of single-sided strategies such that $\sum_i (N_i + T_i)$ has finite initial capital and succeeds on all sequences whose limiting 0-frequency is not 1/2. Hence each computably single-sided random has 0-frequency tending to 1/2. (14)

By '0-frequency' we mean the (relative) frequency of 0 in the initial segments of the sequence. Hence weak s-randomness for $s \in (0, 1)$ does not imply computable single-sided randomness. However in Proposition 2.11, we will see that the left-c.e. version of single-sided randomness does imply weak 1/2-randomness.

2.4 Speed of success and effective Hausdorff dimension

The martingale approach to algorithmic information theory was introduced by [Schnorr, 1971a,b] who also showed some interest in the rate of success of (super)martingales M, and in particular the classes

$$S_h(M) = \left\{ X \mid \limsup_n \frac{M(X\restriction_n)}{h(n)} = \infty \right\}$$

where $h : \mathbb{N} \to \mathbb{N}$ is a computable non-decreasing function. Later [Lutz, 2000, 2003] showed that the Hausdorff dimension of a class of reals can be characterized by the exponential 'success rates' of left-c.e. supermartingales, and in that light defined the effective Hausdorff dimension $\dim(X)$ of a real X as the infimum of the $s \in (0, 1)$ such that $X \in S_h(M)$ for some left-c.e. supermartingale M, where $h(n) = 2^{(1-s)n}$. Then [Mayordomo, 2002] showed that

$$\dim(X) = \liminf_n \frac{C(X\restriction_n)}{n} = \liminf_n \frac{K(X\restriction_n)}{n} \tag{15}$$

where C and K denote the plain and prefix-free Kolmogorov complexity respectively. Reals with effective Hausdorff dimension 1/2 include partially predictable reals (with an imbalance of 0s and 1s) like $Y \oplus \emptyset$ where Y is algorithmically random, as well as random-looking versions of the halting probability like $\sum_{U(\sigma)\downarrow} 2^{-2|\sigma|}$ for certain universal prefix-free machines U from [Tadaki, 2002]. Martin-Löf random reals have effective dimension 1, but the converse does not hold. Moreover there are computably random reals of effective dimension 0. For more on algorithmic dimension see [Downey and Hirschfeldt, 2010, Chapter 13].

Monotonous betting on sequences with dimension less than half. We construct a computable mixture of separable strategies, which succeeds on every sequence of effective Hausdorff dimension $< 1/2$. For this task we need a characterization of effective dimension in terms of tests. Given $s \in (0, 1)$, an s-test is a uniformly c.e. sequence (V_i) of sets of strings such that $\sum_{\sigma \in V_k} 2^{-s|\sigma|} < 2^{-k}$ for each k. As mentioned in [Downey and Hirschfeldt, 2010, §13.6] and is the case for most notions of effective statistical tests,

$$\text{given } s \in (0, 1) \text{ one can effectively obtain an effective list of all } s\text{-tests.} \tag{16}$$

Since $s < 1$, the condition $\sum_{\sigma \in V_k} 2^{-s|\sigma|} < 2^{-k}$ means that the length of each string in V_k is more than k. These observations will be used in the proof of Theorem 2.11. Let us say that X is *weakly s-random* if it avoids all s-tests (V_i), in the sense that there are only finitely many i such that X has a prefix in V_i. By [Tadaki, 2002], X being weakly s-random is equivalent to $\exists c\ \forall n\ K(X\restriction_n) > s \cdot n - c$. Then by (15),

$$\dim(X) = \sup\{s \mid X \text{ is weakly } s\text{-random}\} \tag{17}$$

which is crucial for the proof of the following fact, which complements our main theorems in §3 and §4.

Theorem 2.11 (Monotonous betting for low dimension)**.** *There exist uniformly computable 0-sided and 1-sided strategies* (N_i) *and* (T_i) *respectively such that the mixture* $\sum_i(N_i + T_i)$ *has finite initial capital and succeeds on all* X *such that* $\dim(X) < 1/2$.

Proof. By (14) that we established in §2.3, it suffices to construct a computable family (N_i) of 0-sided strategies such that $\sum_i N_i$ has finite initial capital and succeeds on every sequence X which has limiting 0-frequency $1/2$ and $\dim(X) < 1/2$. For each X with these properties, by (17) there exists a rational $q < 1/2$ and a q-test (V_i) such that X has prefixes in infinitely many V_i. It suffices to prove that:

$$\text{given } \epsilon > 0,\ q < 1/2 \text{ and a } q\text{-test } (V_i)\text{, we can effectively define a computable family } (M_i) \text{ of 0-sided strategies such that } \textstyle\sum_i M_i \text{ has initial capital less than } \epsilon \text{ and succeeds on every } X \text{ with limiting 0-frequency equal to } 1/2 \text{ and prefixes in infinitely many members of } (V_i). \tag{18}$$

Indeed, given (18) and (16) we can effectively produce a family of 0-sided strategies whose mixture has bounded initial capital and deal with any possible q-test (V_i) for any choice of $q < 1/2$. For the proof of (18), given $\epsilon > 0$, $q < 1/2$ and a q-test (V_i), let k_ϵ be the least integer such that $2^{-k_\epsilon} < \epsilon/2$. We define a computable family (N_σ) of 0-sided strategies (indexed by strings) and let

$$M_i = \sum_{\sigma \in V_{k_\epsilon + i}} N_\sigma \quad \text{and} \quad M = \sum_i M_i.$$

Under this definition of M_i and the properties q-tests, for $M(\lambda) < \epsilon$ it suffices to let $N_\sigma(\lambda) = 2^{-q|\sigma|}$ so that

$$M_i(\lambda) = \sum_{\sigma \in V_{k_\epsilon + i}} N_\sigma(\lambda) = \sum_{\sigma \in V_{k_\epsilon + i}} 2^{-q|\sigma|} < 2^{-k_\epsilon - i} \Rightarrow M(\lambda) < 2 \cdot 2^{-k_\epsilon} < \epsilon.$$

For each i and each σ strategy N_σ starts with $N_\sigma(\lambda) = 2^{-q|\sigma|}$ and bets all capital on all the 0s of σ, while placing no bets on all other strings. Formally, for each non-empty ρ define:

$$N_\sigma(\rho) = \left\{ \begin{array}{ll} N_\sigma(\hat{\rho}) & \text{if } \hat{\rho} * 0 \not\preceq \sigma \\ 2 \cdot N_\sigma(\hat{\rho}) & \text{otherwise.} \end{array} \right\} \quad \text{where } \hat{\rho} \text{ denotes the predecessor of } \rho.$$

Since each N_σ is 0-sided, M_i and M are also 0-sided and, as noted above, $M(\lambda) < \epsilon$. Hence for (18) it remains to verify that M succeeds on every X with limiting 0-frequency $1/2$ and prefixes in infinitely many members of (V_i). To this end we observe that, as a direct consequence of the definitions of N_σ, M_i, M:

$$\text{if } \sigma \in V_{k_\epsilon + i} \text{ has } z_\sigma \text{ many 0s then for each } \rho \succeq \sigma,\ M_i(\rho) \geq N_\sigma(\rho) = N_\sigma(\sigma) = 2^{z_\sigma - q|\sigma|}. \tag{19}$$

Given X with limiting 0-frequency $1/2$ and prefixes in infinitely many members of (V_i), let q' be a rational in $(q, 1/2)$. Since the limiting frequency of 0s in X is $1/2$, there exists some n_0 such that for each $n > n_0$ the number $z_{X\upharpoonright_n}$ of 0s in $X \upharpoonright_n$ is more than $q'n$, so $2^{z_{X\upharpoonright_n} - qn} > 2^{(q'-q)n}$. Given any constant c, let $n_1 > n_0$ be such that $2^{(q'-q)n_1} > c$. Let $n_2 > \max\{n_1, k_\epsilon\}$ be such that X has a prefix in V_{n_2} and all strings in V_{n_2} are of length at least n_1. If σ is a prefix of X in V_{n_2}, by the definition of M and (19), for all $n \geq n_2$ we have

$$M(X \upharpoonright_n) \geq M_{n_2}(X \upharpoonright_n) \geq N_\sigma(\sigma) \geq 2^{z_\sigma - q} > 2^{(q'-q)n_1} > c.$$

Since c was arbitrary, this shows that that $\lim_n M(X \upharpoonright_n) = \infty$ for all X with the properties of (18). □

3 The power of single-sided martingales and their mixtures

We show that if a computable martingale succeeds on some casino sequence X, then there exists a computable single-sided martingale which succeeds on X. This is a consequence of the following decomposition, which was also noticed independently by Frank Stephan.

Lemma 3.1 (Single-sided decomposition)**.** *Every martingale M is the product of a 0-sided martingale N and a 1-sided martingale T. Moreover N, T are computable from M.*

Proof. For ease of notation we let $M_\sigma, N_\sigma, T_\sigma$ denote $M(\sigma), N(\sigma), T(\sigma)$ respectively. Let $\sigma \to w_M(\sigma)$ be the wagers of M and let $N_\lambda = T_\lambda = \sqrt{M_\lambda}$. Define the wagers of N, T respectively by:

$$w_N(\sigma) = \left\{ \begin{array}{cl} w_M(\sigma)/T_\sigma & \text{if } w_M(\sigma) < 0,\, T_\sigma > 0; \\ 0 & \text{otherwise}; \end{array} \right\}, \quad w_T(\sigma) = \left\{ \begin{array}{cl} w_M(\sigma)/N_\sigma & \text{if } w_M(\sigma) > 0,\, N_\sigma > 0; \\ 0 & \text{otherwise}; \end{array} \right\}$$

We show by induction that $M_\sigma = N_\sigma \cdot T_\sigma$ for all σ. By definition this holds for $\sigma = \lambda$; suppose that it holds for σ. If $w_M(\sigma) = 0$ then $w_N(\sigma) = w_T(\sigma) = 0$ so $M_\rho = N_\rho \cdot T_\rho$ holds for the immediate successors ρ of σ. If $w_M(\sigma) < 0$ then $w_T(\sigma) = 0$ so $T_{\sigma*i} = T_\sigma$ and

$$M_{\sigma*1} = M_\sigma + w_M(\sigma) = N_\sigma \cdot T_\sigma + w_M(\sigma) = T_\sigma \cdot (N_\sigma + w_N(\sigma)) = T_{\sigma*1} \cdot N_{\sigma*1}.$$

In the same way we have $M_{\sigma*0} = T_{\sigma*0} \cdot N_{\sigma*0}$. The case where $w_M(\sigma) > 0$ is entirely symmetric. □

Corollary 3.2. *Given a computable martingale M, there exist a 0-sided martingale N_0 and a 1-sided martingale N_1 such that for each X on which M is successful, at least one of N_0, N_1 is successful.*

Corollary 3.2 is a direct consequence of Lemma 3.1. and says that, in terms of computable strategies, if there exists a successful strategy against the casino, there exists a successful single-sided strategy. This fact is no longer true for mixtures or strongly left-c.e. martingales (recall the equivalence from §2.2).

Theorem 3.3 (Mixtures of single-sided martingales)**.** *There exists a real of effective Hausdorff dimension 1/2 such that no single-sided (or separable) strongly left-c.e. martingale succeeds on it.*

It is instructive to contrast Theorem 3.3 with Proposition 2.11. Note that for each rational $s \in (0, 1)$ there are reals X with effective Hausdorff dimension s with computable subsequences, so that single-sided strategies succeed easily on them. The basic idea for proving Theorem 3.3 is most clearly demonstrated by proving the following simpler statement, which only deals with a single separable strategy:

> Given the mixture M of a computable family (M_i) of separable martingales M, there exists a left-c.e. real of effective Hausdorff dimension 1/2 such that M does not succeed on it. (20)

The proof of (20) is a computable construction of the approximation to the required real X, and is presented in §3.1–§3.3 in a modular way, so that it can be used in the more involved proof of Theorem 3.3. The only issue that separates the proof of (20) form the proof of Theorem 3.3 is the lack of universality and effective lists of martingales that was discussed in §2.2.

3.1 Idea and plan for the proof of (20)

We will construct the real X of (20) so as to extend a sequence of initial segments (σ_n). Given M as in (20) we will construct X of effective Hausdorff dimension 1/2 such that $M(X \upharpoonright_n)$ is bounded above. Without loss of generality we can assume that $M(\lambda) < 2^{-1}$. In order to ensure the dimension requirement for X, it suffices to ensure that

$$K_V(\sigma_n) \le |\sigma_n| \cdot q_n \tag{21}$$

for all n, where V is a prefix-free machine that we also construct, K_V is the Kolmogorov complexity with respect to V, and (q_n) is a computable decreasing sequence of rationals tending to 1/2. Let us set

$$q_n = 1/2 + 3/(n+2) \quad \text{and} \quad \hat{M}(\sigma) = \max_{n \le |\sigma|} M(\sigma \restriction_n) \tag{22}$$

We will ensure that for all n:

$$\hat{M}(\sigma_n) \le 2^{-1} + \sum_{i<n} 2^{-i-2}. \tag{23}$$

One way to think about this requirement is to try to ensure that $\hat{M}(\sigma_n) - \hat{M}(\sigma_{n-1}) \le 2^{-n-1}$ for all n. Supposing inductively that σ_{n-1} has been determined, the task of keeping $\hat{M}(\sigma_n) - \hat{M}(\sigma_{n-1})$ small potentially involves changing the approximation to σ_n a number of times, since M is a left-c.e. martingale. This instability of the final value of σ_n is in conflict with (21). The main idea for handling this conflict is that if we choose σ_n from a collection of strings which have roughly similar number of 0s and 1s, then a single-sided strategy is limited to winning on around half of the available bits. With such a limitation on the components of M, the separability of M ensures that the growth potential of M is also limited, in a way that allows the satisfaction of (21). Since the construction deals with approximations (N_i) of M, it is crucial for this argument that the intermediate bets $N_t - N_s$ between two stages $s < t$ are single-sided, or separable. As we discussed in §2.1, such an approximation can be chosen when M is a mixture of a computable family (M_i) of separable strategies.

The next concern, given the restriction to strings with balanced number of 0s and 1s, is to be able to choose an extension of σ_n where capital does not increase substantially (note that without the restriction to a particular set of extensions of σ_n, we can choose an extension where the capital does not increase at all).

Lemma 3.4 (Low capital gain somewhere)**.** *Given any σ, any $\delta > 0$ and any set S of extensions of σ of some fixed length $|\sigma| + n$ such that $|S| \ge (1-\delta) \cdot 2^n$, there exists at least one string $\tau \in S$ such that $M(\tau^*) \le M(\sigma)/(1-\delta)$ for all τ^* with $\sigma \subseteq \tau^* \subseteq \tau$.*

Proof. Towards a contradiction suppose that there exists no such string in S, and for each $\tau \in S$ let τ^* be the shortest initial segment extending σ for which $M(\tau^*) > M(\sigma)/(1-\delta)$. Then $S^* = \{\tau^* \mid \tau \in S\}$ is a prefix-free set of strings. Since every element of S has an initial segment in S^* it follows that:

$$\sum_{\tau^* \in S^*} 2^{-|\tau^*|} \cdot M(\tau^*) > (1-\delta) \cdot 2^{-|\sigma|} \cdot \frac{M(\sigma)}{1-\delta} = 2^{-|\sigma|} \cdot M(\sigma)$$

which contradicts Kolmogorov's inequality relative to σ. □

Note that $M(\sigma)/(1-\delta) = M(\sigma) + M(\sigma) \cdot \delta/(1-\delta)$, so a small multiplicative amplification of the capital from σ to τ can be translated into a small additive increase in $M(\tau) - M(\sigma)$, as long as we keep $M(\sigma)$ under a fixed bound. Once σ_{n-1} has been chosen and a 'fat' (i.e. high probability) set of appropriate extensions has been determined, Lemma 3.4 tells us that we will be able to choose σ_n without increasing the capital of M by too much. The following fact follows from Lemma 3.4 and the law of large numbers in Lemma 2.10.

σ_n	the nth initial segment of X with approximations $\sigma_n[s]$
s_n	length of σ_n according to the calculations in §3.2
ϵ_n	appropriate value of the error ϵ of Lemmata 3.5 and 3.6 at level n, set as 2^{-n-5}
q_n	bound on $K_V(X \upharpoonright_{s_n})/s_n$ set at $1/2 + 3/(n+2)$
2^{-p_n}	sufficient upper bound on $M(\sigma_{n-1}) - M_t(\sigma_{n-1})$ for $\sigma_n[t] = \sigma_n$ (assuming $\sigma_{n-1}[t] = \sigma_{n-1}$)

Table 1: Parameters for the proof of (20)

Lemma 3.5 (Special extension)**.** *There exists a computable function f such that if M is a non-negative martingale such that $M(\lambda) \leq 1$, then for each $\epsilon \in (0,1)$, $\sigma \in 2^{<\mathbb{N}}$, and $\ell > f(\epsilon)$ there exists $\tau \succeq \sigma$ of length ℓ such that $M(\rho) \leq M(\sigma)/(1-\epsilon)$ for all $\rho \in [\sigma, \tau]$ and the number of zeros (and hence, 1s) in τ after σ is in $((1-\epsilon)(|\tau| - |\sigma|)/2, (1+\epsilon)(|\tau| - |\sigma|)/2)$.*

We show that the potential for success for a separable strategy along the extensions of Lemma 3.5 is limited.

Lemma 3.6 (Growth along special extensions)**.** *Let (N_i), (T_j) be computable families of 0-sided and 1-sided martingales respectively, with finite total initial capital and consider the mixture $M = \sum_i N_i + \sum_j T_j$ with the approximations $M_s = \sum_{i<s} N_i + \sum_{j<s} T_j$. Given ϵ, σ, if τ is the extension of σ of Lemma 3.5 applied on M_s, then for all $t > s$ if $M_t(\sigma) - M_s(\sigma) < 2^{-p}$ then $M_t(\tau) \leq M_s(\tau) + 2^{\delta\cdot(|\tau|-|\sigma|)-p}$, where $\delta := (1+\epsilon)/2$.*

Proof. For simplicity let $N^* = \sum_{i\in[s,t)} N_i$ and $T^* = \sum_{i\in[s,t)} T_i$, so that $M_t(\tau) \leq M_s(\tau) + N^*(\tau) + T^*(\tau)$, and note that N^* is a 0-sided martingale while T^* is a 1-sided martingale. Between stages s and t there is at most 2^{-p} increase in $M(\sigma)$, so $N^*(\sigma) + T^*(\sigma) \leq 2^{-p}$. By the properties of τ, letting $\delta := (1+\epsilon)/2$, there exist at most $\delta \cdot (|\tau| - |\sigma|)$ many 0s between σ and τ and the same is true of the 1s. Hence, since N^*, T^* are single-sided, we have $N^*(\tau) \leq N^*(\sigma) \cdot 2^{\delta\cdot(|\tau|-|\sigma|)}$ and $T^*(\tau) \leq T^*(\sigma) \cdot 2^{\delta\cdot(|\tau|-|\sigma|)}$. By adding these two, using the fact that $N^*(\sigma) + T^*(\sigma) \leq 2^{-p}$, we get $M_t(\tau) \leq M_s(\tau) + 2^{-p} \cdot 2^{\delta\cdot(|\tau|-|\sigma|)}$. □

3.2 Fixing the parameters for the proof of (20)

Let (M_s) be a canonical approximation to M. We will use Lemma 3.6 for the definition of the sequence (σ_i) that we discussed in §3.1. For the approximations to σ_n with $n > 0$, we will apply Lemma 3.6 for the specific values $\epsilon_n = 2^{-n-5}$ of ϵ and p_n of p (to be defined below), thus obtaining increasingly better bounds for larger n. For each n the segment σ_n as well as its approximations will have a fixed length s_n which we motivate and define as follows. Suppose that $n > 0$ and our choice of σ_{n-1} has settled, but that now we are forced to choose a new value of σ_n, because the capital on some initial segment has increased by too much. What does Lemma 3.6 tell us about the increase in capital, 2^{-p_n} say, that must have seen at σ_{n-1} in order for this to occur? A bound for p_n gives a corresponding bound on the number of times that σ_n will have to be chosen: after σ_{n-1} has settled the approximation to the next initial segment σ_n can change at most 2^{p_n} many times. Overall, σ_n can then change at most $2^{\sum_{i<n} p_i} \cdot 2^{p_n}$ many times, and in order to satisfy (21), at each of these changes we need to enumerate to the machine V a description of length $q_n \cdot s_n$. In order to keep the weight of these requests bounded, we will aim at keeping the total weight of the requests corresponding to σ_n bounded above by 2^{-n}, for which it is sufficient that:

$$2^{-s_n q_n} \cdot 2^{p_n} \cdot 2^{\sum_{i<n} p_i} < 2^{-n} \iff 2^{p_n - s_n q_n} < 2^{-n-\sum_{i<n} p_i} \iff s_n q_n - p_n > n + \sum_{i<n} p_i. \tag{24}$$

By the bound given in Lemma 3.6 in order for the growth of $\hat{M}(\sigma_n)$ at each length s_n to be bounded above by 2^{-n-2}, we need to set:

$$p_n = s_n \cdot \delta_n + n + 2 \quad \text{where } \delta_n := (1 + \epsilon_n)/2. \tag{25}$$

By Lemma 3.6 it then follows that any growth of $M(\tau)$ by at least 2^{-n-2} for some τ with $\sigma_{n-1} \subseteq \tau \subseteq \sigma_n$, requires an increase of at least 2^{-p_n} in $M(\sigma_{n-1})$. Then $p_n - q_n s_n = n + 2 + s_n \cdot (\delta_n - q_n)$. By the definitions of q_n, ϵ_n we have $\delta_n < q_n$, so (24) reduces to:

$$s_n \cdot (q_n - \delta_n) > 2n + 2 + \sum_{i<n} p_i \iff s_n \geq \frac{2n + 2 + \sum_{i<n} p_i}{q_n - \delta_n}.$$

Considering the bound of Lemma 3.5 for the existence of a special extension of σ_{n-1}, it suffices that:

$$s_n = \max\left\{\frac{2n + 2 + \sum_{i<n} p_i}{q_n - \delta_n}, f(\epsilon_n)\right\}. \tag{26}$$

where f is the computable function of Lemma 3.5.

3.3 Construction and verification for (20)

We inductively define the approximations $\sigma_n[s]$ of σ_n for all n, in stages s. Let $\sigma_0[s] = \lambda$ for all s, $s_0 = 0$ and $\delta_n := (1 + \epsilon_n)/2$ for all n. The following notion incorporates the properties of Lemma 3.5 in the framework of the construction and the specific values of the parameters that were set in §3.2.

Definition 3.7 (Special extensions)**.** At each stage $s + 1$ and for each $n > 0$ such that $\sigma_{n-1}[s] \downarrow$ we say that τ is a special extension of $\sigma_{n-1}[s]$ if $|\tau| = s_n$, $\hat{M}_s(\tau) \leq \hat{M}(\sigma_{n-1})[s]/(1 - \epsilon_n)$ and the number of 0s as well as the number of 1s between $\sigma_{n-1}[s]$ and τ is at most $\delta_n \cdot (s_n - s_{n-1})$.

Definition 3.8 (Attention)**.** At stage $s + 1$ the segment σ_n requires attention if $n > 0$ and either $\sigma_n[s] \downarrow$ and $\hat{M}_{s+1}(\sigma_n[s]) > 2^{-1} + \sum_{i<n} 2^{-i-2}$, or $\sigma_n[s] \uparrow$.

Construction for (20). At stage $s + 1$ pick the least $n \leq s$ such that σ_n requires attention, if such exists. If $\sigma_n[s] \uparrow$, define $\sigma_n[s + 1]$ to be the leftmost special extension of $\sigma_{n-1}[s]$. If $\sigma_n[s] \downarrow$, set $\sigma_i[s + 1] \uparrow$ for all $i \geq n$. In any case, let $k \leq s$ the least (if such exists) such that $\sigma_k[s + 1] \downarrow$ and $K_{V_s}(\sigma_k[s + 1]) > q_k \cdot s_k$, and issue a V-description of $\sigma_k[s + 1]$ of length $q_k \cdot s_k$.

Remark. If at stage $s + 1$ segment $\sigma_n[s + 1]$ is newly defined as a special extension of σ_{n-1} we have

$$\hat{M}(\sigma_n)[s + 1] \leq \hat{M}_{s+1}(\sigma_{n-1}[s])/(1 - \epsilon_n) = \hat{M}_{s+1}(\sigma_{n-1}[s]) + \hat{M}_{s+1}(\sigma_{n-1}[s]) \cdot \epsilon_n/(1 - \epsilon_n). \tag{27}$$

Since $\hat{M}(\sigma_{n-1}) < 1$ and $\epsilon_n/(1 - \epsilon_n) < 2^{-n-2}$, condition (27) implies

$$\hat{M}(\sigma_n)[s + 1] \leq \hat{M}(\sigma_{n-1})[s + 1] + 2^{-n-2}. \tag{28}$$

Verification of the construction for (20). By Lemma 3.4 we can always find a special extension as required in the first clause of the construction. In this sense, the construction of $(\sigma_n[s])$ is well-defined. In any interval of stages where σ_{n-1} remains defined, successive values of σ_n are lexicographically increasing. It follows that each $\sigma_n[t]$ converges to a final value σ_n such that $\sigma_n \prec \sigma_{n+1}$. The real X determined by the initial segments σ_n is thus left-c.e. and since (27) implies (28), we have $\hat{M}(X \upharpoonright_n) < 1$ for all n.

It remains to show that the weight of V is bounded above by 1. Suppose that σ_n gets newly defined at stage $s+1$ and at stage $t > s+1$ it becomes undefined, while $\sigma_{n-1}[j] \downarrow$ for all $j \in [s, t]$. Then $\hat{M}(\sigma_n)[s+1] \leq \hat{M}(\sigma_{n-1})[s] + 2^{-n-2}$. Since σ_n becomes undefined at stage t, we have $\hat{M}_t(\sigma_n[s+1]) > 2^{-1} + \sum_{i<n} 2^{-i-2}$. By Lemma 3.6 and (25) it follows that $\hat{M}_t(\sigma_{n-1}[s+1]) - M(\sigma_{n-1})[s+1] > 2^{-p_n}$. Hence:

during an interval of stages where σ_{n-1} remains defined, σ_n can take at most 2^{p_n} values

which means that the weight of the V-descriptions that we enumerate for strings of length s_n is at most $2^{-s_n q_n} \cdot 2^{\sum_{i \leq n} p_i}$. By the definition of s_n in (26) and (24) this weight is bounded above by 2^{-n}. So the total weight of the descriptions that are enumerated into V is at most 1.

4 Proof of Theorem 3.3 and generalizations

It is possible to adapt the proof of Lemma (20) into an effective construction for the proof of Theorem 3.3, which also gives that the real X is left-c.e. For simplicity, we opt for a less constructive initial segment argument for the proof of Theorem 3.3, which uses the facts we obtained in §3 in a modular way. The price we pay is that the constructed X is no-longer left-c.e. as in (20). The following is the main tool for the proof of Theorem 3.3, where q_n has the same value as in §3.

Lemma 4.1 (Inductive property)**.** *There exists a prefix-free machine V such that for each $n > 0$, $\sigma_0 \prec \cdots \prec \sigma_{n-1}$ and $M = \sum_{j<x} N_j$, where each N_j, $j < x$ is a mixture of a computable family of strictly single-sided martingales with*

$$\hat{M}(\sigma_{n-1}) < 2^{-1} + \sum_{i<n-1} 2^{-i-2} \tag{29}$$

where σ_0 is the empty string, there exists $\sigma_n \succ \sigma_{n-1}$ such that

$$K_V(\sigma_n) < |\sigma_n| \cdot q_n \quad \text{and} \quad \hat{M}(\sigma_n) \leq \left(2^{-1} + \sum_{i<n-1} 2^{-i-2}\right) + 2^{-n-2}. \tag{30}$$

4.1 Proof of Theorem 3.3 from Lemma 4.1

Let $(M_j[s])$ be a (non-effective) list of all canonical approximations to left-c.e. martingales M_j with initial capital < 1. This list includes an approximation to each strongly left-c.e. strictly single-sided martingale, and by Lemma 2.6 it suffices to show the theorem with regard to the martingales in this list. Using Lemma 4.1 we inductively define a sequence (σ_i) of strings such that $\sigma_i \prec \sigma_{i+1}$ for each i, σ_0 is the empty string, and (30) holds for each $n > 0$ and the separable left-c.e. martingale

$$S_n := \sum_{j<n} 2^{-|\sigma_j|-j-2} \cdot M_j.$$

We will ensure that for each n we have

$$K_V(\sigma_n) < |\sigma_n| \cdot q_n \qquad \text{and} \qquad \hat{S}_n(\sigma_n) \leq 2^{-1} + \sum_{i<n} 2^{-i-2} \leq 1 \tag{31}$$

where V is the prefix-free machine of Lemma 4.1. If we let X to be the real defined by the initial segments (σ_i), then the second clause of (31) implies that for each $j, x \in \mathbb{N}$ we have $M_j(X \upharpoonright_x) \leq 2^{|\sigma_j|+j+2} < \infty$ as required. Hence by the choice of (M_j) and Lemma 2.6, no single-sided left-c.e. martingale succeeds on X and by Theorem 2.11 the effective Hausdorff dimension is at least 1/2. The first clause of (31) implies that the effective Hausdorff dimension of X is at most 1/2. Hence (31) is sufficient for the proof of Theorem 3.3.

For the inductive use of Lemma 4.1 in the construction, consider the following inequality for some n (the equality is always true) which can be thought of as obtained at step n from Lemma 4.1:

$$\hat{S}_n(\sigma_n) \leq \left(2^{-1} + \sum_{i<n-1} 2^{-i-2}\right) + 2^{-n-2} = \left(2^{-1} + \sum_{i<n} 2^{-i-2}\right) - 2^{-n-2} \tag{32}$$

and note that, since $M_n(\sigma_n) < 2^{|\sigma_n|}$ we have $2^{-|\sigma_n|-n-2} \cdot M_n(\sigma_n) < 2^{-n-2}$ so

$$\text{if} \quad (32) \quad \text{holds then} \quad \hat{S}_{n+1}(\sigma_n) \leq 2^{-1} + \sum_{i<n} 2^{-i-2} \tag{33}$$

which is the hypothesis that is needed in order to apply Lemma 4.1 at step $n+1$ which will define σ_{n+1}.

Construction. Let σ_0 be the empty string and let V be the machine from Lemma 4.1. For each $n > 0$, inductively assume that (29) holds for S_{n-1} and let σ_n be an extension of σ_{n-1} such that (30) holds.

Verification. First we show that the construction is well-defined. Note that (30) holds for $n = 0$. Assuming that $n > 0$ and (30) holds with S_n in place of M, by (33) it follows that (29) holds with S_{n+1} in place of M and with $n+1$ in place of n. Hence by Lemma 4.1 at step $n+1$ there exists an extension σ_{n+1} of σ_n which satisfies (30) with $n+1$ in place of n. This concludes the justification that the construction is well-defined. The construction, and in particular condition (30) imposed on the extensions, shows that (31) holds for each n. This concludes the verification of the properties of the constructed sequence (σ_i) and, as discussed above, the proof of Theorem 3.3.

4.2 Preliminaries for the proof of Lemma 4.1

The construction of V of Lemma 4.1 will be computable, so for the proof of the lemma we need to define an effective map which takes as an input η a description (index) of M and strings σ_i, $i < n$, and always outputs a *sufficiently small* part V_η of V (dealing with the specific input η) and an approximation $\sigma_n[s]$ such that

> if the input $\eta = (M, \sigma_i, i < n)$ meets the hypothesis of Lemma 4.1 then $\sigma_n[s]$ converges to some σ_n which satisfies the properties of the lemma with V_η in place of V. (34)

Since Lemma 4.1 asks for a single machine V that applies to all inputs, we need to make sure that the special machines V_η are sufficiently small so that there exists a machine V with the property that K_V is bounded by K_{V_η} for all inputs η. In order to express this property precisely, let $(N_i[s])$ an effective sequence (viewed as a double list of functions $\sigma \mapsto N_i(\sigma)[s]$) of all canonical partial computable approximations of all (partial computable) mixtures of single-sided strategies. The set H of inputs is the set of all tuples $\eta = (i, \sigma_0, \ldots, \sigma_{n-1})$ where $i \in \mathbb{N}$ is interpreted as an index in the list $(N_i[s])$, and $\sigma_0 \prec \cdots \prec \sigma_{n-1}$ is a chain of strings. Let $g : H \to \mathbb{N}$ a one-to-one computable function so that $\sum_{\eta \in H} 2^{-g(\eta)} < 1$.

Given any $\eta \in H$ the map $\eta \mapsto (V_\eta, \sigma_n[s])$ will determine a prefix-free machine V_η such that

$$\text{for each } \eta \in H, \quad \texttt{wgt}(V_\eta) < 2^{-g(\eta)} \qquad \text{so} \qquad \textstyle\sum_{\eta \in H} \texttt{wgt}\left(V_\eta\right) < 1. \tag{35}$$

By (35) we may define a *union* prefix-free machine V such that $K_V(\rho) \le K_{V_\eta}(\rho)$ for each ρ and each $\eta \in H$. We have shown that

> for the proof of Lemma 4.1 it suffices to define a computable map $\eta \mapsto (V_\eta, \sigma_n[s])$ from H to pairs of prefix-free machines and approximations of strings, such that (34) and (35) hold.

The construction of the effective map $\eta \mapsto (V_\eta, \sigma_n[s])$ is a modification of the proof of (20) in §3.2,§3.3 and uses Lemma 3.6 in the same way. The parameters q_n, ϵ_n, p_n are as defined in §2.2. Since here we have a special upper bound $2^{-g(\eta)}$ for the weight of V_η, we need to re-calculate a suitable value for s_n, which is the required lower bound for the length of σ_n of Lemma 4.1. Following §3.2, condition (24) becomes

$$2^{-s_n q_n} \cdot 2^{p_n} < 2^{-g(\eta)} \iff s_n q_n - p_n > g(\eta). \tag{36}$$

Given the definition of p_n in (25) and arguing as in §3.2, in order for the growth of $\hat{M}(\sigma_n)$ at each length s_n to be bounded above by 2^{-n-3}, for (36) it suffices that

$$s_n \cdot (q_n - \delta_n) > g(\eta) + n + 3 \iff s_n \ge \frac{g(\eta) + n + 3}{q_n - \delta_n}$$

so it suffices to define the length of each approximation to σ_n in the proof of Lemma 4.1 by:

$$s_n = s_n(\eta) = \max\left\{\frac{g(\eta) + n + 3}{q_n - \delta_n}, f(\epsilon_n)\right\}. \tag{37}$$

We also need the following simplified version of Definition 3.7 which will be used in the construction.

Definition 4.2 (Special extensions)**.** At stage $s+1$ we say that τ is a special extension of σ_{n-1} if $|\tau| = s_n$ and it satisfies the properties of Lemma 3.5 for $\epsilon := \epsilon_n$, $\sigma := \sigma_{n-1}$, $p := p_n$ and $s+1$ in place of s.

It remains to define and verify the construction of the map $\eta \mapsto (V_\eta, \sigma_n[s])$.

4.3 Construction and verification for the proof of Lemma 4.1

Given $\eta \in H$, let σ_j, $j < n$ be the associated strings in η in order of magnitude. For simplicity, let (M_s) be the canonical partial computable left-c.e. approximation given by η and let $U := V_\eta$. The following construction, on input η produces an effective enumeration U_s of the prefix-free machine $U := V_\eta$ and an effective approximation $\sigma_n[s]$ of the string σ_n (which may or may not converge).

Construction of U, σ_n from η. At stage 0 we let $\sigma_n[0] \uparrow$ and U_0 be empty. At stage $s+1$, do the following provided that M_{s+1} is defined on all strings of length s, and (29) holds at stage $s+1$ (otherwise go to the next stage). Check if one of the following holds:

(i) $\sigma_n[s] \uparrow$, there have been at most 2^{p_n} previous definitions of σ_n in previous stages, and there exists a special extension of σ_{n-1}.

(ii) $\sigma_n[s] \downarrow$ and $\hat{M}_{s+1}(\sigma_n[s]) > 2^{-1} + \sum_{i \in [j, n-1)} 2^{-i-2} + 2^{-n-3}$.

If (i) holds, define $\sigma_n[s+1]$ to be the leftmost special extension of σ_{n-1} as per Definition 4.2. If (ii) holds, set $\sigma_n[s+1] \uparrow$. In any case, if $\sigma_n[s+1] \downarrow$ and $K_{U_s}(\sigma_n[s+1]) > q_n \cdot |\sigma_n[s+1]|$, issue a U-description of $\sigma_n[s+1]$ of length $q_n \cdot |\sigma_n[s+1]|$.

Remark. If $\sigma_n[s+1]$ is newly defined as a special extension of σ_{n-1}, by Definition 4.2 we have that

$$\hat{M}_{s+1}(\sigma_n)[s+1] \le \hat{M}_s(\sigma_{n-1})/(1-\epsilon_n) = \hat{M}_{s+1}(\sigma_{n-1}) + \hat{M}_{s+1}(\sigma_{n-1})\cdot \epsilon_n/(1-\epsilon_n). \tag{38}$$

By (29) referenced at stage $s+1$, we have $\hat{M}_{s+1}(\sigma_{n-1}) < 1$ so by $\epsilon_n/(1-\epsilon_n) < 2^{-n-3}$ condition (38) implies

$$\hat{M}_{s+1}(\sigma_n[s+1]) \le \hat{M}_{s+1}(\sigma_{n-1}) + 2^{-n-3}. \tag{39}$$

Verification of the constructing of U, σ_n from η.

First we show that the weight of U is bounded above by $g(\eta)$. In this argument we do not assume anything about the input η, the associated approximation (M_s), or the convergence of the approximations $(\sigma_n[s])$. Clause (i) of the construction enforces that

$$\text{the approximation to } \sigma_n \text{ can change at most } 2^{p_n} \text{ many times.} \tag{40}$$

This is the assumption we used in our calculations of (36) and (37), which we can now use to derive the bound on the weight of U based on the values of $p_n, s_n, q_n, \epsilon_n$ that we set. By (40) and since $|\sigma_n[s]| = s_n$, the weight of the U-descriptions that we enumerate for the approximations to σ_n is at most $2^{-s_n q_n}\cdot 2^{p_n}$. The definition of s_n in (37) and (36) imply that the above bound is at most $2^{-g(\eta)}$ as required.

It remains to show that in the case that if (M_s) is a total computable canonical approximation to a single-sided mixture M such that (29) holds, the construction will produce an approximation $\sigma_n[s]$ which converges to a string σ_n after finitely many stages, such that the second clause of (30) holds. Suppose that σ_n gets (re)defined at stage $s+1$ and at stage $t > s+1$ it becomes undefined. By (39) and (29) we have

$$\hat{M}_{s+1}(\sigma_n[s+1]) \le 2^{-1} + \sum_{i\in[j,n-1)} 2^{-i-2} + 2^{-n-3}. \tag{41}$$

Since σ_n becomes undefined at stage t, we have $\hat{M}_t(\sigma_n[s+1]) > 2^{-1} + \sum_{i<n-1} 2^{-i-2} + 2^{-n-2}$. By (39),(41), (29) and an application of Lemma 3.6 for $\epsilon := \epsilon_n$, $p := p_n$, $\sigma := \sigma_{n-1}$ and $\tau := \sigma_n[s+1]$, it follows that $\hat{M}_t(\sigma_{n-1}) - M_{s+1}(\sigma_{n-1}) > 2^{-p_n}$. Since the latter event can occur at most 2^{p_n} many times, we have shown that if σ_n is newly defined by the construction at some stage $s+1$ and this is the 2^{p_n}-th such definition during the construction, then it will never be undefined again, i.e. $\sigma_n[t]\downarrow$ for all $t > s$. In particular, the second clause of (i) in the construction (regarding the number of previous definitions of σ_n) can never block the redefinition of σ_n, subject to the other two condition holding. Given this fact, and Lemma 3.5 which concerns the existence of special extensions, it is not possible that σ_n is undefined for co-finitely many stages; in other words, *for each s_0 there exists $s > s_0$ such that $\sigma_n[s]\downarrow$*. Since (M_s) is a left-c.e. approximation, by (29) and (39) successive values of $\sigma_n[s]$ during redefinitions of σ_n will be lexicographically increasing, so $\sigma_n[s]$ converges to a final value σ_n such that $\sigma_{n-1} \prec \sigma_n$. Since (38) implies (39), we have that the second part of (30) holds, as required. This concludes the proof of (34) and (35) hence, as explained in §4.2, the proof of Lemma 4.1.

4.4 Generalization to decidably-sided strategies

We adapt argument of §4.1-§4.3 in order to prove the following analogue of Theorem 3.3.

Theorem 4.3. *There exists a real X of effective Hausdorff dimension $1/2$ such that no decidably-sided strongly left-c.e. martingale (or mixture of a computable family of f-sided strategies for some computable f) succeeds on it.*

We need the following an analogue of Lemma 4.1 for decidably-sided left-c.e. martingales.

Lemma 4.4 (Inductive property for decidably-sided sums)**.** *There exists a prefix-free machine V with the property that for each $n > 0$, chain of strings $\sigma_0 \prec \cdots \prec \sigma_{n-1}$, computable prediction functions $f_i, i < n$ and $M = \sum_{i<n} M_i$, where each $M_i, i < n$ is a mixture of a computable family of f_i-decidably sided martingales with canonical approximation such that*

$$\hat{M}(\sigma_{n-1}) < 2^{-1} + \sum_{i \in [j, n-1)} 2^{-i-2}$$

where σ_0 is the empty string, there exists $\sigma_n \succ \sigma_{n-1}$ such that

$$K_V(\sigma_n) < |\sigma_n| \cdot q_n \quad \text{and} \quad \hat{M}(\sigma_n) \leq \left(2^{-1} + \sum_{i \in [j, n-1)} 2^{-i-2}\right) + 2^{-n-2}.$$

Theorem 4.3 follows from Lemma 4.4 by the argument of §4.1 which derived Theorem 3.3 from Lemma 4.1. The only difference is that here $(M_j[s])$ is a list of all canonical approximations to mixtures of decidably-sided martingales whose initial capital is less than 1. For the proof of Lemma 4.4 we need to obtain analogues of the key facts from §3 for the case of decidably-sided martingales. We start with the following analogue of Lemma 3.5, which follows by a direct application of Lemma 3.4 to the law of large numbers in Lemma 2.10, applied to the intersection of finitely many events.

Lemma 4.5 (Special extensions for decidably sided)**.** *There exists a computable g such that for each $\epsilon \in (0, 1)$, $\sigma \in 2^{<\mathbb{N}}$, $n > 0$, and (M_j, f_j), $j < n$ where each f_j is a prediction function and M_j is an f_j-sided martingale with $M_j(\lambda) \leq 1$, and each $\ell > g(\epsilon, n)$, there exists $\tau \succeq \sigma$ of length ℓ such that for each $j < n$,*

$$\text{the number of correct } f_j\text{-predictions in } [\sigma, \tau] \text{ is in } \Big((1-\epsilon)(|\tau|-|\sigma|)/2, (1+\epsilon)(|\tau|-|\sigma|)/2\Big)$$

and $M_j(\rho)[s] \leq M_j(\sigma)[s]/(1-\epsilon)$ for all ρ with $\sigma \subseteq \rho \subseteq \tau$.

Now we may obtain the required analogue of Lemma 3.6.

Lemma 4.6 (Growth along special extension for decidably sided)**.** *Let (M_j, f_j), $j < n$ be as in Lemma 4.5, let $M_j[s]$ be canonical approximations of M_j, and define $N := \sum_{j<n} M_j$ and $N_s := \sum_{j<n} M_j[s]$. Given $\epsilon > 0, p, s \in \mathbb{N}, \sigma \in 2^{<\omega}$, if τ is the extension of σ given by Lemma 4.5, then for all $t > s$:*

$$N_t(\sigma) - N_s(\sigma) < 2^{-p} \Rightarrow N_t(\tau) \leq N_s(\tau) + 2^{\delta \cdot |\tau| - p}$$

where $\delta := (1 + \epsilon)/2$.

The proof of Lemma 4.6 for the special case where N is itself a mixture of f-sided martingales for a computable f is entirely analogous to the proof of Lemma 3.6 which refers to single-sided martingales, with the difference that Lemma 4.5 is used in place of Lemma 3.5. The case where N is the sum of finitely many such mixtures (with distinct prediction functions f_j) follows from the special case in the same way that the separable case of Lemma 3.6 follows from the special case of a single-sided martingale (recall the first paragraph of the proof of Lemma 3.6).

It remains to show that a straightforward adaptation of the argument in §4.2, §4.3 proves Lemma 4.4. The entire set-up of §4.2 remains the same, including the parameter values and Definition 4.2 which is later used in the construction, with the exception that instead of Lemma 3.5 we use Lemma 4.6. The construction of the required map in §4.3 remains exactly the same, except that the updated version of Definition 4.2 of special extensions is used (based on Lemma 4.6 instead of Lemma 3.6). The verification of the construction in §4.3 also remains the same, except that the reference to Lemma 3.6 is replaced with a reference to Lemma 4.6. This concludes the proof of Lemma 4.4 and, as explained above, the proof of Theorem 4.3.

5 Conclusion and some questions

We have studied the strength of monotonous strategies, which bet constantly on the same outcome (single-sided martingales) or bet on a computable outcome (decidably-sided martingales). In the case of computable strategies we have seen that they are as strong as the unrestricted strategies, while in the case of uniform effective mixtures of strategies (strongly left-c.e. martingales) they are significantly weaker. On the other hand, for casino sequences of effective Hausdorff dimension less than 1/2, successful left-c.e. strategies can be replaced by successful uniform effective mixtures of single-sided strategies.

Limitations of the present work and open problems. Our main negative results, Theorems 3.3 and 4.3, rely on two main properties: (a) the given strategies are martingales and not merely supermartingales; (b) the given monotonous martingales are not merely left-c.e. but strongly l,c.e., i.e. are assumed to have left-c.e. wagers. Restriction (a) relates to the non-interchangeability between martingales and supermartingales under monotonousness, as discussed in §2.2; the main interest on (a) is the connection with a problem of Kastermans, which we briefly discuss below. Perhaps most significant is restriction (b), which rests on the difference between *mixtures of computable families of monotonous strategies* on the one hand, and *monotonous mixtures of computable families of strategies* on the other. The difference in these two approaches of combining monotonousness with computable enumerability of strategies, described as uniform and non-uniform in (i), (ii) of §1 respectively, relies on whether the *intermediate bets* witnessed by a computable observer with access to the approximation of the strategy are monotonous or not. Our main open question is whether (b) is essential for Theorems 3.3 and 4.3:

> **Question:** If a left-c.e. martingale succeeds on X, does there exist a left-c.e. single-sided strategy (i.e. a single-sided martingale M which is the mixture of a computable family of strategies) which succeeds on X? (42)

Equivalently, we can ask if the standard notion of algorithmic randomness, Martin-Löf randomness, can be defined with respect to single-sided left-c.e. martingales. A third limitation (c) in Theorem 4.3 is the assumption, included in Definition 2.3, that the prediction functions f are *total computable* and not merely partial computable, allowing the possibility of partiality on states σ where the wager is 0. Such a generalization would formalize a notion of *partially decidably-sided* strategies, which cannot be dealt with by the argument in the proof of Theorem 4.3.

Relation to a problem of Kastermans. Consider the case of left-c.e. supermartingales that are partially decidably-sided, according to the above discussion; such strategies are known as *kastergales*, see [Downey and Hirschfeldt, 2010, §7.9]). Kastermans, as reported in [Downey, 2012] and [Downey and Hirschfeldt, 2010, §7.9] asked whether there exists a sequence where all kastergales are bounded, but some computably enumerable strategy succeeds. A simple negative answer to this question would be that for every real X

where a left-c.e. martingale succeeds, there exists a single-sided, or even just decidably-sided martingale which succeeds on X. First, note that a positive answer to (42) would give a very simple and elegant negative answer to Kastermans' question. In the same fashion, Theorem 4.3 can be viewed as a partial negative answer to Kastermans' question. Then limitations (a), (b) and (c) of our methods discussed above are the obstacles in extending our partial answer to a full negative answer to Kastermans' question.

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