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Topological stable rank of $\mathcal{E}'(\mathbb{R})$

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Abstract
The set $\mathcal{E}'(\mathbb{R})$ of all compactly supported distributions, with the operations of addition, convolution, multiplication by complex scalars, and with the strong dual topology is a topological algebra. In this article, it is shown that the topological stable rank of $\mathcal{E}'(\mathbb{R})$ is 2.

Keywords K-theory · Compactly supported distributions

Mathematics Subject Classification Primary 46F10; Secondary 19B10 · 19K99 · 46H05

1 Introduction

The aim of this article is to show that the topological stable rank (a notion from topological K-theory, recalled below) of $\mathcal{E}'(\mathbb{R})$ is 2, where $\mathcal{E}'(\mathbb{R})$ is the classical topological algebra of compactly supported distributions, with the strong dual topology $\beta(\mathcal{E}', \mathcal{E})$, pointwise vector space operations, and convolution taken as multiplication.

We recall some key notation and facts about $\mathcal{E}'(\mathbb{R})$ in Sect. 2 below, including its strong dual topology $\beta(\mathcal{E}', \mathcal{E})$, and in Sect. 3, we will recall the notion of topological stable rank of a topological algebra.

We will prove our main result, stated below, in Sects. 4 and 5.

Theorem 1.1 *Let $\mathcal{E}'(\mathbb{R})$ be the algebra of all compactly supported distributions on $\mathbb{R}$, with*

- *pointwise addition, and pointwise multiplication by complex scalars,*
- *convolution taken as the multiplication in the algebra, and*
- *the strong dual topology $\beta(\mathcal{E}', \mathcal{E})$.*

Then the topological stable rank of $\mathcal{E}'(\mathbb{R})$ is equal to 2.

2 The topological algebra $\mathcal{E}'(\mathbb{R})$

For background on topological vector spaces and distributions, we refer to [2,6,7,11–13,16].

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Let $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R})$ be the space of functions $\varphi : \mathbb{R} \to \mathbb{C}$ that are infinitely many times differentiable. We equip $\mathcal{E}(\mathbb{R})$ with the topology of uniform convergence on compact sets for the function and its derivatives. This is defined by the following family of seminorms: for a compact subset K of $\mathbb{R}$, and $M \in \{0, 1, 2, 3 \cdots\} = \mathbb{Z}_{\geq 0}$, we define

$$p_{K,M}(\varphi) = \sup_{0\leq m\leq M} \sup_{x\in K} |\varphi^{(m)}(x)| \quad \text{for } \varphi \in \mathcal{E}(\mathbb{R}).$$

The space $\mathcal{E}(\mathbb{R})$ is

- metrizable,
- a Fréchet space, and
- a Montel space;

see e.g. [7, Example 3, p.239].
By a topological algebra, we mean the following:

Definition 2.1 (*Topological algebra*) A complex algebra $\mathcal{A}$ is called a *topological algebra* if it is equipped with a topology $\mathcal{T}$ making the following maps continuous, with the product topologies on the domains:

- $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto a + b \in \mathcal{A}$
- $\mathbb{C} \times \mathcal{A} \ni (\lambda, a) \mapsto \lambda \cdot a \in \mathcal{A}$
- $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto ab \in \mathcal{A}$

We equip the dual space $\mathcal{E}'(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$ with the strong dual topology $\beta(\mathcal{E}', \mathcal{E})$, defined by the seminorms

$$p_B(T) = \sup_{\varphi\in B} |\langle T, \varphi\rangle|,$$

for bounded subsets B of $\mathcal{E}(\mathbb{R})$. Then $\mathcal{E}'(\mathbb{R})$, being the strong dual of the Montel space $\mathcal{E}(\mathbb{R})$, is a Montel space too [12, 5.9, p. 147]. This has the consequence that a sequence in $\mathcal{E}'(\mathbb{R})$ is convergent in the $\beta(\mathcal{E}', \mathcal{E})$ topology if and only if it is convergent in the weak dual/weak-$*$ topology $\sigma(\mathcal{E}', \mathcal{E})$ of pointwise convergence on $\mathcal{E}(\mathbb{R})$; see e.g. [16, Corollary 1, p. 358].

As usual, let $\mathcal{D}(\mathbb{R})$ denote the space of all compactly supported functions from $C^\infty(\mathbb{R})$, and $\mathcal{D}'(\mathbb{R})$ denote the space of all distributions. The vector space $\mathcal{E}'(\mathbb{R})$ can be identified with the subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all distributions having compact support. If $\mathcal{D}'(\mathbb{R})$ is also equipped with its strong dual topology, then one has a continuous injection $\mathcal{E}'(\mathbb{R}) \hookrightarrow \mathcal{D}'(\mathbb{R})$. For $T, S \in \mathcal{E}'(\mathbb{R})$, we define their convolution $T * S \in \mathcal{E}'(\mathbb{R})$ by

$$\langle T * S, \varphi\rangle = \Big\langle T,\ \big[\, x \mapsto \big\langle S, \varphi(x + \cdot)\big\rangle \,\big] \Big\rangle, \quad \varphi \in \mathcal{E}(\mathbb{R}).$$

The map $* : \mathcal{E}'(\mathbb{R}) \times \mathcal{E}'(\mathbb{R}) \to \mathcal{E}'(\mathbb{R})$ is (jointly) continuous; see for instance [13, Chapter VI, §3, Theorem IV, p. 157].

Thus $\mathcal{E}'(\mathbb{R})$, endowed with the strong dual topology, forms a topological algebra with pointwise vector space operations, and with convolution taken as multiplication. The multiplicative identity element is δ_0, the Dirac delta distribution supported at 0. In general, we will denote by δ_a the Dirac delta distribution supported at $a \in \mathbb{R}$.

We also recall that the Fourier–Laplace transform of a compactly supported distribution $T \in \mathcal{E}'(\mathbb{R})$ is an entire function, given by

$$\widehat{T}(z) = \Big\langle T,\ \big(x \mapsto e^{-2\pi i x z}\big) \Big\rangle \quad (z \in \mathbb{C}),$$

see e.g. [16, Proposition 29.1, p. 307].

3 Topological stable rank

An analogue of the Bass stable rank (useful in algebraic K-theory) for topological rings, called the topological stable rank, was introduced in the seminal article [10].

Definition 3.1 (*Unimodular tuple, Topological stable rank*)
Let $\mathcal{A}$ be a commutative unital topological algebra with multiplicative identity element denoted by 1, endowed with a topology $\mathcal{T}$.
We define $\mathcal{A}^n := \mathcal{A} \times \cdots \times \mathcal{A}$ (n times), endowed with the product topology.

- (Unimodular n-tuple) Let $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. We call an n-tuple $(a_1, \ldots, a_n) \in \mathcal{A}^n$ *unimodular* if there exists an n-tuple $(b_1, \ldots, b_n) \in \mathcal{A}^n$ such that the Aryabhatta-Bézout equation $a_1 b_1 + \cdots + a_n b_n = 1$ is satisfied. The set of all unimodular n-tuples is denoted by $U_n(\mathcal{A})$. Note that $U_1(\mathcal{A})$ is the group of invertible elements of $\mathcal{A}$. An element from $U_2(\mathcal{A})$ is referred to as a *coprime* pair. It can be seen that if $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, then $U_{n+1}(\mathcal{A})$ is dense in $\mathcal{A}^{n+1}$.
- (Topological stable rank) If there exists a least natural number $n \in \mathbb{N}$ for which $U_n(\mathcal{A})$ is dense in $\mathcal{A}^n$, then that n is called the *topological stable rank* of $\mathcal{A}$, denoted by tsr $\mathcal{A}$. If no such n exists, then tsr$\mathcal{A}$ is said to be infinite.

While the notion of topological stable rank was introduced in the context of *Banach* algebras, the above extends this notion in a natural manner to topological algebras. The topological stable rank of many concrete Banach algebras has been determined previously in several works (e.g. [5,14,15]). In this article, we determine the topological stable rank of the classical topological algebra $\mathcal{E}'(\mathbb{R})$ from Schwartz's distribution theory.

4 tsr$(\mathcal{E}'(\mathbb{R})) \geq 2$

The idea is that if tsr$(\mathcal{E}'(\mathbb{R}))$ were 1, then we could approximate any T from $\mathcal{E}'(\mathbb{R})$ by compactly supported distributions whose Fourier transform would be zero-free, and by an application of Hurwitz Theorem, $\widehat{T}$ would need to be zero-free too, which gives a contradiction, since we can easily choose T at the outset to not allow this.

Proposition 4.1 tsr$(\mathcal{E}'(\mathbb{R})) \geq 2$.

Proof Suppose on the contrary that tsr$(\mathcal{E}'(\mathbb{R})) = 1$. Let

$$T = \frac{\delta_{-1} - \delta_1}{2i} \in \mathcal{E}'(\mathbb{R}).$$

By our assumption, $U_1(\mathcal{E}'(\mathbb{R}))$ is dense in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$. But then the set $U_1(\mathcal{E}'(\mathbb{R}))$ is also sequentially dense: This is a consequence of the fact that a subset F of $\mathcal{E}'(\mathbb{R})$ is closed in $\beta(\mathcal{E}', \mathcal{E})$ if and only if it is sequentially closed. (See [9, Satz 3.5, p. 231], which says that E', with the $\beta(E', E)$-topology, is sequential whenever E is Fréchet–Montel. A locally convex space F is *sequential* if any subset of F is closed if and only if it is sequentially closed. If F has this property, then the closure of any subset equals its sequential closure, and therefore being dense is the same as being sequentially dense. In our case, $E = \mathcal{E}(\mathbb{R})$ is Fréchet–Montel, and so $\mathcal{E}'(\mathbb{R})$ is sequential. In fact, in the remark following [9, Satz 3.5], the case of $\mathcal{E}'(\mathbb{R})$ is mentioned as a corollary.)

Thus there exists a sequence $(T_n)_{n\in\mathbb{N}}$ in $U_1(\mathcal{E}'(\mathbb{R}))$ such that $T_n \stackrel{n\to\infty}{\longrightarrow} T$ in $\mathcal{E}'(\mathbb{R})$. But since each T_n is invertible in $\mathcal{E}'(\mathbb{R})$, there exists a sequence $(S_n)_{n\in\mathbb{N}}$ in $\mathcal{E}'(\mathbb{R})$ such that

$$T_n * S_n = \delta_0 \quad \text{for all } n \in \mathbb{N}.$$

Taking the Fourier–Laplace transform, we obtain

$$\widehat{T_n}(z) \cdot \widehat{S_n}(z) = 1 \quad \text{for all } z \in \mathbb{C} \text{ and all } n \in \mathbb{N}.$$

In particular, the entire functions $\widehat{T_n}$ are all zero-free.

But as $T_n \stackrel{n\to\infty}{\longrightarrow} T$ in $\mathcal{E}'(\mathbb{R})$, we now show that $(\widehat{T_n})_{n\in\mathbb{N}}$ converges to $\widehat{T}$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. The pointwise convergence of $(\widehat{T_n})_{n\in\mathbb{N}}$ to $\widehat{T}$ is clear by taking the test function $x \mapsto e^{-2\pi i x z}$:

$$\widehat{T_n}(z) = \langle T_n, e^{-2\pi i z \cdot}\rangle \stackrel{n\to\infty}{\longrightarrow} \langle T, e^{-2\pi i z \cdot}\rangle = \widehat{T}(z).$$

Now for any $\varphi \in \mathcal{E}(\mathbb{R})$, we know that the sequence $(\langle T_n, \varphi\rangle)_{n\in\mathbb{N}}$ converges to $\langle T, \varphi\rangle$, and in particular, the set

$$\Gamma(\varphi) := \{\langle T_n, \varphi\rangle : n \in \mathbb{N}\}$$

is bounded, for every $\varphi \in \mathcal{E}(\mathbb{R})$. By the Banach–Steinhaus Theorem for Fréchet spaces (see for example [11, Theorem 2.6, p. 45]), applied in our case to the Fréchet space $\mathcal{E}(\mathbb{R})$, we conclude that

$$\Gamma = \{T_n : n \in \mathbb{N}\}$$

is equicontinuous. Thus for every $\epsilon > 0$, there exists a neighbourhood V of 0 in $\mathcal{E}(\mathbb{R})$ such that $T_n(V) \subset B(0, \epsilon) := \{z \in \mathbb{C} : |z| < \epsilon\}$ for all $n \in \mathbb{N}$. From here it follows that there exist $M \in \mathbb{Z}_{\geq 0}$, $R > 0$ and $C > 0$ such that

$$|\langle T_n, \varphi\rangle| \leq C\Big(1 + \sup_{0\leq m\leq M} \sup_{|x|\leq R} |\varphi^{(m)}(x)|\Big).$$

By taking $\varphi = (x \mapsto e^{-2\pi i x z})$ in the above, we obtain

$$|\widehat{T_n}(z)| \leq C'(1 + |z|)^{M'} e^{R'|z|}, \quad z \in \mathbb{C},\; n \in \mathbb{N}.$$

Also, by the Payley–Wiener–Schwartz Theorem [2, Theorem 4.12, p. 139] for $T \in \mathcal{E}'(\mathbb{R})$, we have

$$|\widehat{T}(z)| \leq C''(1 + |z|)^{M''} e^{R''|z|}, \quad z \in \mathbb{C},\; n \in \mathbb{N}.$$

It now follows that for some constants C_*, M_*, R_* that

$$|\widehat{T_n}(z) - \widehat{T}(z)| \leq C_*(1 + |z|)^{M_*} e^{R_*|z|}, \quad z \in \mathbb{C},\; n \in \mathbb{N}.$$

But this means that the pointwise convergent sequence $(\widehat{T_n})_{n\in\mathbb{N}}$ of entire functions is uniformly bounded on compact subsets of $\mathbb{C}$ (that is, the sequence constitutes a normal family). Then it follows from Montel's Theorem (see e.g. [17, Exercise 9.4, p. 157]) that $(\widehat{T_n})_{n\in\mathbb{N}}$ converges to $\widehat{T}$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$.

But now by Hurwitz Theorem (see e.g. [17, Exercise 5.6, p.85]), and considering, say, the compact set $K = \{z \in \mathbb{C} : |z| \leq 1\}$, we conclude that $\widehat{T}$ must be either be identically zero on K or that it must be zero-free in K. But $\widehat{T}$ is neither:

$$\widehat{T}(z) = \frac{e^{2\pi i z} - e^{-2\pi i z}}{2i} = \sin(2\pi z),$$

a contradiction. Hence $\text{tsr}(\mathcal{E}'(\mathbb{R})) \geq 2$. □

An alternative Proof of Proposition 4.1, suggested by Peter Wagner, is as follows. The theorem of supports ([6, Theorem 4.3.3]) implies that $U_1(\mathcal{E}'(\mathbb{R}))$ equals the set of nonzero multiples of δ_a for arbitrary $a \in \mathbb{R}$, and this set is not dense in $\mathcal{E}'(\mathbb{R})$. We give the details below. First, one can show the following structure result for $U_1(\mathcal{E}'(\mathbb{R}))$.

Proposition 4.2 $U_1(\mathcal{E}'(\mathbb{R})) = \{c\delta_a : a \in \mathbb{R},\ 0 \neq c \in \mathbb{C}\}$.

Proof It is clear that $\{c\delta_a : a \in \mathbb{R},\ 0 \neq c \in \mathbb{C}\} \subset U_1(\mathcal{E}'(\mathbb{R}))$ since

$$(c\delta_a) * (c^{-1}\delta_{-a}) = \delta_0.$$

Now suppose that $T \in U_1(\mathcal{E}'(\mathbb{R}))$. Then there exists an $S \in \mathcal{E}'(\mathbb{R})$ such that $T * S = \delta_0$. By the Theorem on Supports [6, Theorem 4.3.3, p. 107], we have

$$\text{c.h.supp}(T * S) = \text{c.h.supp}(T) + \text{c.h.supp}(S),$$

where, for a distribution $E \in \mathcal{E}'(\mathbb{R})$, the notation $\text{c.h.supp}(E)$ is used for the closed convex hull of $\text{supp}(E)$, that is, the intersection of all closed convex sets containing $\text{supp}(E)$. So we obtain

$$\{0\} = \text{c.h.supp}(\delta_0) = \text{c.h.supp}(T * S) = \text{c.h.supp}(T) + \text{c.h.supp}(S),$$

from which it follows that $\text{c.h.supp}(T) = \{a\}$ and $\text{c.h.supp}(S) = \{-a\}$ for some $a \in \mathbb{R}$. But then also $\text{supp}(T) = \{a\}$ and $\text{supp}(S) = \{-a\}$. As distributions with support in a point p are linear combinations of the Dirac delta distribution δ_p and its derivatives $\delta_p^{(n)}$ [16, Theorem 24.6, p. 266], we conclude that S and T have the form

$$T = \sum_{n=0}^{N} t_n \delta_a^{(n)},$$

$$S = \sum_{m=0}^{M} s_m \delta_{-a}^{(m)},$$

for some integers $N, M \geq 0$ and some complex numbers t_n, s_m $(0 \leq n \leq N, 0 \leq m \leq M)$. Now $T * S = \delta_0$ implies that $N = M = 0$ and $t_0 s_0 = 1$, thanks to the linear independence of the set

$$\{\delta_{-a}, \delta'_{-a}, \delta''_{-a}, \ldots\} \cup \{\delta_0, \delta'_0, \delta''_0, \ldots\} \cup \{\delta_a, \delta'_a, \delta''_a, \ldots\}$$

in the complex vector space $\mathcal{E}'(\mathbb{R})$. In particular $t_0 \neq 0$. Thus

$$T = t_0\delta_a \in \{c\delta_p : p \in \mathbb{R},\ 0 \neq c \in \mathbb{C}\}.$$

Consequently, $U_1(\mathcal{E}'(\mathbb{R})) = \{c\delta_a : a \in \mathbb{R},\ 0 \neq c \in \mathbb{C}\}$. □

Based on the above, we can now give the following alternative proof of Proposition 4.1.

Proof We show $U_1(\mathcal{E}'(\mathbb{R}))$ is not dense in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$. If it were, then it would be sequentially dense too, and so for each element T of $\mathcal{E}'(\mathbb{R})$, there would exist a sequence in $U_1(\mathcal{E}'(\mathbb{R}))$ that converges to T in the $\beta(\mathcal{E}', \mathcal{E})$ topology, and hence also in the $\sigma(\mathcal{E}', \mathcal{E})$

topology. But we now show that $\delta_0' \in \mathcal{E}'(\mathbb{R})$ cannot be approximated in the $\sigma(\mathcal{E}', \mathcal{E})$ topology by elements from $U_1(\mathcal{E}'(\mathbb{R})) = \{c\delta_a : a \in \mathbb{R},\ 0 \neq c \in \mathbb{C}\}$. Suppose, on the contrary, that $(c_n\delta_{a_n})_{n\in\mathbb{N}}$ converges to δ_0' in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$.

We first note that $(a_n)_{n\in\mathbb{N}}$ is bounded. For if not, then there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $|a_{n_k}| > 2$ for all $k \in \mathbb{N}$. Now choose a $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi'(0) = 1$ and $\varphi \equiv 0$ on $\mathbb{R} \setminus (-1, 1)$. Then we arrive at the contradiction that

$$0 = c_{n_k} \cdot 0 = c_{n_k} \cdot \varphi(a_{n_k}) = \langle c_{n_k}\delta_{a_{n_k}}, \varphi\rangle \stackrel{k\to\infty}{\longrightarrow} \langle \delta_0', \varphi\rangle = -\varphi'(0) = -1.$$

So $(a_n)_{n\in\mathbb{N}}$ is bounded.

Next we show that $(c_n)_{n\in\mathbb{N}}$ converges to 0. Let $R > 0$ be such that $|a_n| < R$ for all $n \in \mathbb{N}$. Let $\psi \in \mathcal{D}(\mathbb{R})$ be such that $\psi \equiv 1$ on $[-R, R]$. Then we have

$$c_n = c_n \cdot 1 = c_n \cdot \psi(a_n) = \langle c_n\delta_{a_n}, \psi\rangle \stackrel{n\to\infty}{\longrightarrow} \langle \delta_0', \psi\rangle = -\psi'(0) = 0.$$

Finally, we show that $(c_n\delta_{a_n})_{n\in\mathbb{N}}$ converges to 0 in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$. For any $\chi \in \mathcal{D}(\mathbb{R})$, we have

$$|\langle c_n\delta_{a_n}, \chi\rangle| = |c_n| \cdot |\chi(a_n)| \le |c_n| \cdot \|\chi\|_\infty \stackrel{n\to\infty}{\longrightarrow} 0 \cdot \|\chi\|_\infty = 0.$$

So $(c_n\delta_{a_n})_{n\in\mathbb{N}}$ converges to 0 in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$. But this is a contradiction, since $0 \neq \delta_0'$ in $\mathcal{E}'(\mathbb{R})$. Consequently, $U_1(\mathcal{E}'(\mathbb{R}))$ is not dense in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$, and so $\text{tsr}(\mathcal{E}'(\mathbb{R})) \ge 2$. □

5 tsr($\mathcal{E}'(\mathbb{R})$) $\le$ 2

The idea is to reduce the determination of $\text{tsr}(\mathcal{E}'(\mathbb{R}))$ to $\text{tsr}(\mathbb{C}[z])$ of the polynomial ring $\mathbb{C}[z]$ as follows. Given a pair from $\mathcal{E}'(\mathbb{R})$, we use mollification to make a pair in $\mathcal{D}(\mathbb{R})$, and then approximate the resulting smooth functions by a linear combination of Dirac distributions with uniform spacing. The uniform spacing affords the identification of the linear combination of Dirac deltas with the ring of polynomials.

For $n \in \mathbb{N}$, we define the collection $\mathbf{D}_n$ of all 'finitely supported Dirac delta combs' with spacing $1/n$ by

$$\mathbf{D}_n := \text{span}\,\{\delta_{k/n} : k \in \mathbb{Z}\},$$

where 'span' means the set of all (finite) linear combinations.

Lemma 5.1 (Approximating a pair of Dirac combs by a *unimodular* pair) *Let $n \in \mathbb{N}$ and $T, S \in \mathbf{D}_n$. Then there exist sequences $(T_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ in $\mathbf{D}_n$, which converge to T, S, respectively, in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$, and hence*[1] *also in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$, and are such that for each k, $(T_k, S_k) \in U_2(\mathcal{E}'(\mathbb{R}))$.*

Proof Write $T = \sum_{\ell=-L}^{L} t_\ell \delta_{\ell/n}$, and $S = \sum_{\ell=-L}^{L} s_\ell \delta_{\ell/n}$, for some $L \in \mathbb{N}$, $t_\ell, s_\ell \in \mathbb{C}$. Define

$$p_T := t_{-L} + t_{-L+1}z + \cdots + t_L z^{2L},$$
$$p_S := s_{-L} + s_{-L+1}z + \cdots + s_L z^{2L}.$$

For a given $k \in \mathbb{N}$, let $\epsilon = 1/(2^k \cdot 2L) > 0$. Then we can perturb the coefficients of the polynomials p_T, p_S within a distance of ϵ to make them have no common zeros, that is after

[1] Because $\mathcal{E}'(\mathbb{R})$ is a Montel space; see [16, Corollary 1, p. 358].

perturbation of coefficients they are coprime in the ring $\mathbb{C}[z]$. Indeed any polynomial p_T, p_S can be factorized as

$$p_T = C\prod(z-\alpha_\ell), \quad p_S = C'\prod(z-\beta_\ell),$$

and if there is some common zero $\alpha_\ell = \beta_{\ell'}$, we simply replace $\beta_{\ell'}$ by $\beta_{\ell'}+\epsilon'$ with an ϵ' small enough so that the final coefficients (of this new perturbed polynomial obtained from p_S), which are polynomial functions of the zeros, lie within the desired ϵ distance of the coefficients of p_S. So we can choose $\widetilde{t}_{-L,k},\dots,\widetilde{t}_{L,k}$ and $\widetilde{s}_{-L,k},\dots,\widetilde{s}_{L,k}$ such that for all $\ell=-L,\dots,L$, we have

$$|t_\ell - \widetilde{t}_{\ell,k}| < \frac{1}{2^k}\cdot\frac{1}{2L} \quad \text{and} \quad |s_\ell - \widetilde{s}_{\ell,k}| < \frac{1}{2^k}\cdot\frac{1}{2L},$$

and so that

$$\begin{aligned}\widetilde{p}_{T,k} &:= \widetilde{t}_{-L,k} + \widetilde{t}_{-L+1,k}z + \dots + \widetilde{t}_{L,k}z^{2L},\\ \widetilde{p}_{S,k} &:= \widetilde{s}_{-L,k} + \widetilde{s}_{-L+1,k}z + \dots + \widetilde{s}_{L,k}z^{2L}\end{aligned}$$

have no common zeros. Thus $\widetilde{p}_{T,k}$, $\widetilde{p}_{S,k}$ are coprime in $\mathbb{C}[z]$, and hence there exist polynomials $q_{T,k}, q_{S,k} \in \mathbb{C}[z]$ ([1, Corollary 8.5, p. 374]) such that

$$\widetilde{p}_{T,k}\cdot q_{T,k} + \widetilde{p}_{S,k}\cdot q_{S,k} = 1.$$

Set $Q_{T,k} := z^L q_{T,k}$ and $Q_{S,k} := z^L q_{S,k}$, and

$$\begin{aligned}P_{T,k} &:= \widetilde{t}_{-L,k}z^{-L} + \widetilde{t}_{-L+1,k}z^{-L+1} + \dots + \widetilde{t}_{L,k}z^{L},\\ P_{S,k} &:= \widetilde{s}_{-L,k}z^{-L} + \widetilde{s}_{-L+1,k}z^{-L+1} + \dots + \widetilde{s}_{L,k}z^{L}.\end{aligned}$$

Then in the ring $\mathbb{C}[z,z^{-1}]$ of linear combinations of monomials z^n, where $n\in\mathbb{Z}$ (i.e. the Laurent polynomial ring $\mathbb{C}[z,z^{-1}] = \mathbb{C}[z,w]/\langle zw-1\rangle$; see for example [1, p. 367]), we have

$$P_{T,k}\cdot Q_{T,k} + P_{S,k}\cdot Q_{S,k} = 1. \tag{1}$$

Suppose that $Q_{T,k}$ and $Q_{S,k}$ have the expansions

$$\begin{aligned}Q_{T,k} &= \tau_{L',k}z^{L+L'} + \tau_{L'-1,k}z^{L+L'-1} + \dots + \tau_{0,k}z^{L},\\ Q_{S,k} &= \sigma_{L',k}z^{L+L'} + \sigma_{L'-1,k}z^{L+L'-1} + \dots + \sigma_{0,k}z^{L}.\end{aligned}$$

Finally, set

$$T_k := \sum_{\ell=-L}^{L}\widetilde{t}_{\ell,k}\delta_{\ell/n}, \quad S_k := \sum_{\ell=-L}^{L}\widetilde{s}_{\ell,k}\delta_{\ell/n},$$

and

$$\begin{aligned}U_k &:= \tau_{L',k}\delta_{(L+L')/n} + \tau_{L'-1,k}\delta_{(L+L'-1)/n} + \dots + \tau_{0,k}\delta_{L/n},\\ V_k &:= \sigma_{L',k}\delta_{(L+L')/n} + \sigma_{L'-1,k}\delta_{(L+L'-1)/n} + \dots + \sigma_{0,k}\delta_{L/n}.\end{aligned}$$

Then it follows from (1) that

$$T_k * U_k + S_k * V_k = \delta_0. \tag{2}$$

To see this, we note that $\Phi : \mathbb{C}[z,z^{-1}] \to \mathbf{D}_n$ given by

$$\Phi(z) = \delta_{1/n} \text{ and } \Phi(1) = \delta_0$$

defines a ring homomorphism, and then (2) above follows by applying Φ on both sides of (1). Hence $(T_k, U_k) \in U_2(\mathcal{E}'(\mathbb{R}))$. Also, for any $\varphi \in \mathcal{E}(\mathbb{R})$, we have

$$\begin{aligned}\left|\langle (T - T_k), \varphi \rangle\right| &= \left|\sum_{\ell=-L}^{L} (t_\ell - \widetilde{t}_{\ell,k}) \langle \delta_{\ell/n}, \varphi\rangle \right| \\ &= \frac{1}{2^k} \cdot \frac{1}{2L} \cdot 2L \cdot \sup_{x \in [-\frac{L}{n}, \frac{L}{n}]} |\varphi(x)| \\ &= \frac{1}{2^k} \cdot \sup_{x \in [-\frac{L}{n}, \frac{L}{n}]} |\varphi(x)| \stackrel{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

Hence $T_k \stackrel{k\to\infty}{\longrightarrow} T$ in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$ as $k \to \infty$. But then this convergence is also valid in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$, by [16, Corollary 1, p. 358], since $\mathcal{E}'(\mathbb{R})$ is a Montel space. Similarly, $S_k \stackrel{k\to\infty}{\longrightarrow} S$ in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$ as $k \to \infty$, and again, the convergence holds in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$. This completes the proof. □

Lemma 5.2 (Approximation in $\mathcal{E}'(\mathbb{R})$ by Dirac combs)
Let $T \in \mathcal{E}'(\mathbb{R})$. Then there exists a sequence $(T_n)_{n\in\mathbb{N}}$ such that

- *for all $n \in \mathbb{N}$, $T_n \in \mathbf{D}_n$, and*
- *$T_n \stackrel{n\to\infty}{\longrightarrow} T$ in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$, and hence also in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$.*

Proof Let $k \in \mathbb{N}$ be such that the support of T is contained in $(-k, k)$. We first produce a mollified approximating sequence for T. Let $\varphi : \mathbb{R} \to [0, \infty)$ be any test function in $\mathcal{D}(\mathbb{R})$ with support in $[-a, a]$ for some $a > 0$, and such that

$$\int_{\mathbb{R}} \varphi(x) dx = 1.$$

Then we know that if we define $\varphi_m(x) := m \cdot \varphi(mx)$ $(m \in \mathbb{N})$, then for each m,

$$f_m := T * \varphi_m$$

is a smooth function having a compact support, and moreover,

$$T * \varphi_m \stackrel{m\to\infty}{\longrightarrow} T$$

in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$; see for example [2, Theorem 3.3, p.97]. So the convergence is also valid in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$. Moreover, as the support of $f_m = T * \varphi_m$ is contained in the sum of the supports of φ_m and of T, for all m large enough, say $m \geq M$, we have

$$\begin{aligned}\text{supp}(T * \varphi_m) &\subset \text{supp}(T) + \text{supp}(\varphi_m) \\ &\subset \text{supp}(T) + [-a/m, a/m] \\ &\subset [-k, k].\end{aligned}$$

From now on, we will assume that $m \geq M$, so that $\text{supp}(f_m) \subset [-k, k]$. Now we will approximate f_m by Dirac comb elements. To this end, we define

$$T_{m,n} := \sum_{\ell=0}^{n-1} \frac{2k}{n} \cdot f_m\Big(-k + \frac{2k}{n}\ell\Big) \cdot \delta_{-k+\frac{2k}{n}\ell} \quad \in \mathbf{D}_n.$$

We will show that $T_{m,n} \stackrel{n\to\infty}{\longrightarrow} f_m$ in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$. Let $\psi \in \mathcal{E}(\mathbb{R})$. Then

$$\begin{aligned}\langle T_{m,n}, \psi\rangle &= \left\langle \sum_{\ell=0}^{n-1} \frac{2k}{n} \cdot f_m\Big(-k + \frac{2k}{n}\ell\Big) \cdot \delta_{-k+\frac{2k}{n}\ell}, \; \psi \right\rangle \\ &= \sum_{\ell=0}^{n-1} \frac{2k}{n} \cdot f_m\Big(-k + \frac{2k}{n}\ell\Big) \Big\langle \delta_{-k+\frac{2k}{n}\ell}, \; \psi \Big\rangle \\ &= \sum_{\ell=0}^{n-1} \frac{2k}{n} \cdot f_m\Big(-k + \frac{2k}{n}\ell\Big) \cdot \psi\Big(-k + \frac{2k}{n}\ell\Big).\end{aligned}$$

Thus $\langle T_{m,n}, \psi\rangle$ gives a Riemann sum for the integral of the continuous function $f_m\psi$ with compact support contained in $[-k, k]$, giving

$$\begin{aligned}&\big|\langle T_{m,n}, \psi\rangle - \langle f_m, \psi\rangle\big| \\ &= \left|\sum_{\ell=0}^{n-1} \frac{2k}{n} \cdot f_m\Big(-k+\frac{2k}{n}\ell\Big) \cdot \psi\Big(-k+\frac{2k}{n}\ell\Big) - \int_{-k}^{k} f_m(x)\psi(x)dx\right| \stackrel{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Hence $T_{m,n} \stackrel{n\to\infty}{\longrightarrow} f_m$ in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$. As $\mathcal{E}'(\mathbb{R})$ is a Montel space, this convergence is also valid in $(\mathcal{E}'(\mathbb{R}), \beta(\mathcal{E}', \mathcal{E}))$, and the proof is completed. □

Proposition 5.3 $\operatorname{tsr}(\mathcal{E}'(\mathbb{R})) \le 2$.

Proof Let $T, S \in \mathcal{E}'(\mathbb{R})$. Throughout this proof, $\mathcal{E}'(\mathbb{R})$ is endowed with the strong dual topology $\beta(\mathcal{E}', \mathcal{E})$, and then $(\mathcal{E}'(\mathbb{R}))^2 = \mathcal{E}'(\mathbb{R}) \times \mathcal{E}'(\mathbb{R})$ is equipped the product topology. Let V be a neighbourhood of (T, S) in $(\mathcal{E}'(\mathbb{R}))^2$. By Lemma 5.2, it follows that $\bigcup_{n\in\mathbb{N}} (\mathbf{D}_n \times \mathbf{D}_n)$ is sequentially dense, and hence dense, in $(\mathcal{E}'(\mathbb{R}))^2$.

Thus there exists a pair $(T_*, S_*) \in V \cap (\mathbf{D}_n \times \mathbf{D}_n)$ for some $n \in \mathbb{N}$. By Lemma 5.1, there exists a sequence $(T_k, S_k)_{k\in\mathbb{N}}$ in $(\mathbf{D}_n \times \mathbf{D}_n) \cap U_2(\mathcal{E}'(\mathbb{R}))$ that converges to (T_*, S_*) in $(\mathcal{E}'(\mathbb{R}))^2$. Since V is also a neighbourhood of (T_*, S_*) in $(\mathcal{E}'(\mathbb{R}))^2$, there exists an index K large enough so that for all $k > K$, $(T_k, S_k) \in V$.

Consequently, $U_2(\mathcal{E}'(\mathbb{R}))$ is dense in $(\mathcal{E}'(\mathbb{R}))^2$. □

Proof of Theorem 1.1 It follows from Propositions 4.1 and 5.3 that the topological stable rank of $(\mathcal{E}'(\mathbb{R}), +, \cdot, *, \beta(\mathcal{E}', \mathcal{E}))$ is equal to 2. □

Remarks 5.4 1. From the proofs, it is clear that we have shown that $U_1(\mathcal{E}'(\mathbb{R}))$ is not dense in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$, while $U_2(\mathcal{E}'(\mathbb{R}))$ is sequentially dense, and hence dense, in $(\mathcal{E}'(\mathbb{R}))^2$ endowed with the product topology with $\mathcal{E}'(\mathbb{R})$ bearing the $\sigma(\mathcal{E}', \mathcal{E})$ topology.
However, we note that $* : \mathcal{E}'(\mathbb{R}) \times \mathcal{E}'(\mathbb{R}) \to \mathcal{E}'(\mathbb{R})$ is not continuous if we use the $\sigma(\mathcal{E}', \mathcal{E})$ topology on $\mathcal{E}'(\mathbb{R})$: For example, in $(\mathcal{E}'(\mathbb{R}), \sigma(\mathcal{E}', \mathcal{E}))$, we have that $\delta_{\pm n} \stackrel{n\to\infty}{\longrightarrow} 0$, so that in the product topology on $(\mathcal{E}'(\mathbb{R}))^2$, we have $(\delta_n, \delta_{-n}) \stackrel{n\to\infty}{\longrightarrow} (0, 0)$. But on the other hand, we have $\delta_n * \delta_{-n} = \delta_{n-n} = \delta_0 \stackrel{n\to\infty}{\longrightarrow} \delta_0 \ne 0 = 0 * 0$. So $(\mathcal{E}'(\mathbb{R}), +, \cdot, *, \sigma(\mathcal{E}', \mathcal{E}))$ is not a topological algebra in the sense of our Definition 2.1.

2. We remark that in higher dimensions, with a similar analysis, it can be shown that $\operatorname{tsr}(\mathcal{E}'(\mathbb{R}^d)) \le d + 1$.
3. The Bass stable rank (a notion from algebraic K-theory, recalled below) of $\mathcal{E}'(\mathbb{R})$ is not known.

If $\mathcal{A}$ is a commutative unital ring, then $(a_1, \dots, a_n, b) \in U_{n+1}(\mathcal{A})$ is called *reducible* if there exists an n-tuple $(\alpha_1, \dots, \alpha_n) \in \mathcal{A}^n$ such that $(a_1+\alpha_1 b, \dots, a_n+\alpha_n b) \in U_n(\mathcal{A})$. It can be seen that if every element of $U_{n+1}(\mathcal{A})$ is reducible, then every element of $U_{n+2}(\mathcal{A})$ is reducible too. The *Bass stable rank* of $\mathcal{A}$, denoted by bsr $\mathcal{A}$, is the smallest $n \in \mathbb{N}$ such that every element in $U_{n+1}(\mathcal{A})$ is reducible, and if no such n exists, then bsr $\mathcal{A} := \infty$. It is known that for commutative unital Banach algebras $\mathcal{A}$, bsr $\mathcal{A} \le$ tsr $\mathcal{A}$ [4, Theorem 3]. But the validity of such an inequality in the context of topological algebras does not seem to be known. We conjecture that $\text{bsr}(\mathcal{E}'(\mathbb{R})) = 2$.

4. There are also several other natural convolution algebras of distributions on $\mathbb{R}$, for example

$$\mathcal{D}'_{\geq \cdot}(\mathbb{R}) := \{T \in \mathcal{D}'(\mathbb{R}) : \text{supp}(T) \text{ is bounded on the left}\},$$
$$\mathcal{D}'_{\geq 0}(\mathbb{R}) := \{T \in \mathcal{D}'(\mathbb{R}) : \text{supp}(T) \subset [0, \infty)\},$$

and we leave the determination of the stable ranks of these algebras as open questions.

5. [8, Corollary 3.1] gives a ‘corona-type’ pointwise condition for coprimeness in $\mathcal{E}'(\mathbb{R})$, reminiscent of the famous Carleson corona condition[2] of coprimeness in the Banach algebra $H^\infty(\mathbb{D})$:
$T_1, T_2 \in U_2(\mathcal{E}'(\mathbb{R}))$ if and only if there exist positive C, N, M such that for all numbers $z \in \mathbb{C}, \quad |\widehat{T_1}(z)| + |\widehat{T_2}(z)| \geq C(1 + |z|^2)^{-N} e^{-M|\text{Im}(z)|}$.

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[2] The Hardy algebra $H^\infty(\mathbb{D})$ is the Banach algebra of all bounded and holomorphic functions on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The Carleson Corona Theorem [3] says that $(f_1, f_2) \in U_2(H^\infty(\mathbb{D}))$ if and only if there exists a $\delta > 0$ such that for all $z \in \mathbb{D}$, $|f_1(z)| + |f_2(z)| > \delta$.

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