Beyond Haar and Cameron-Martin: the Steinhaus support

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In memoriam Herbert Heyer 1936-2018

Abstract. Motivated by a Steinhaus-like interior-point property involving the Cameron-Martin space of Gaussian measure theory, we study a group-theoretic analogue, the Steinhaus triple (H, G, μ), and construct a Steinhaus support, a Cameron-Martin-like subset, $H(\mu)$ in any Polish group G corresponding to 'sufficiently subcontinuous' measures μ, in particular for 'Solecki-type' reference measures.

Key-words. Cameron-Martin space, Gaussian measures, relativized Steinhaus (interior-point) property, Steinhaus triple, Steinhaus support, amenability at 1, relative quasi-invariance, abstract Wiener space, measure subcontinuity.

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1 Introduction

For many purposes, one needs a *reference measure.* In discrete situations such as the integers $\mathbb{Z}$, one has *counting measure.* In Euclidean space $\mathbb{R}^d$, one has *Lebesgue measure.* In locally compact groups, one has *Haar measure.* In infinite-dimensional settings such as Hilbert space, one has neither local compactness nor Haar measure. Here various possibilities arise, some pathological – which we avoid by restricting to *Radon* measures (below). One is to use Christensen's concept of *Haar-null sets* (below), even though there is no Haar measure; see [Chr1,2], Solecki [Sol], and the companion paper to this, [BinO7]. Another is to use *Gaussian measures;* for background see e.g. Bogachev [Bog1,3], Kuo [Kuo] and for Gaussian processes, Lifshits [Lif], Marcus and Rosen [MarR], Ibragimov and Rozanov [IbrR]. Another is to weaken the invariance of measure to (relative) *quasi-invariance* – see [Bog2, §9.11, p. 304-305], and also §9.3 and §9.16 below.

Hilbert spaces are rather special, and the natural setting for Christensen's Haar-null sets is Banach spaces. The Banach and Hilbert settings combine (or intertwine) in Gross's concept of *abstract Wiener space,* where (identifying a Hilbert space H with its dual) one has a triple $B \subseteq H \subseteq B^{**}$, with both inclusions continuous dense embeddings. (It is tempting, but occasionally misleading, to speak of embedded 'subspaces' despite either subset here typically having a topology *finer* than the *subspace* topology induced by the containing space, hence the use of quotation marks.) This is (essentially) the setting of *reproducing-kernel Hilbert spaces (RKHS);* see e.g. Berlinet and Thomas-Agnon [BerTA]. Crucial here is the Cameron-Martin(-Maruyama-Girsanov) theorem ([CamM1,2,3], [Gir]; [Bog1, 2.4], [Bog3, 1.4]). Here a suitable *translation* gives a change of measure, the two measures being equivalent, with Radon-Nikodym derivative given by the *Cameron-Martin formula,* (CM) below. Crucial also are the *Gaussian dichotomy* results (two Gaussian measures on the same space are either equivalent or mutually singular). One has equivalence under translation exactly when the translator is in the *Cameron-Martin space* [Bog1, 2.2]; these are the *admissible translators*. We note that the Girsanov *change of measure* (by translation, using (CM)) is the key to, e.g., Black-Scholes theory in mathematical finance (for background see e.g. [BinK]).

Our purpose here is to construct a group-theoretic analogue of the Cameron-Martin space arising in Gaussian measure theory. We are motivated by a *relativization of the Steinhaus interior-point property* [Ste], to be introduced below (important to classical regular variation – see e.g. [BinGT, Th. 1.1.1]), i.e. with the notion of interior relativized to a distinguished subset equipped with a finer topology: in brief a 'relativized version'. Though the classical paradigm may fail in an infinite-dimensional Hilbert space, it can nevertheless hold relative to an embedded (necessarily, compactly so – see the final assertion of Th. 3.4) space. Such is precisely the case when interiors are taken relative to the Cameron-Martin space.

We recall the Gaussian context in a locally compact topological group G. For simplicity, take G Euclidean. Then matters split, according to the support of the Gaussian measure. If this is the whole of G, the measure has a density (given by the classical and familiar Edgeworth formula for the multi-normal (multi-variate normal) distribution, see e.g. [BinF, 4.16]). If not, the measure is singular viewed on G, but becomes non-singular when restricted to the subgroup generated by its support (see §9.12). (This situation is familiar in statistics: behaviour may seem degenerate only because it is viewed in a context bigger than its natural one; see e.g. [BinF].) Another instance of a similar 'support-degeneracy' phenomenon arises in the Itô-Kawada theorem, when a (suitably non-degenerate) probability measure μ has its convolution powers converging to Haar measure on a subgroup F of G, the closed subgroup generated by its support [Hey1, § 2.1]. In each case, the moral is the obvious one: if one begins in the wrong context, identify the right one and start again.

Let X be a locally convex topological vector space; it suffices for us to take X a separable *Fréchet* space (that is, having a translation-invariant complete metric). Equip this with a Gaussian (probability) measure γ ('gamma for Gaussian', following [Bog1, Ch. 2] and §9.10; for Radon Gaussian measures in this context see [Bog1, Ch.3]). Suppose further that the dual satisfies $X^* \subseteq L^2(\gamma)$. Write $\gamma_h(K) := \gamma(K+h)$ for the translate by h. *Relative quasi-invariance* of γ_h and γ, that for all compact K

$$\gamma_h(K) > 0 \text{ iff } \gamma(K) > 0,$$

holds relative to a set of vectors $h \in X$ (the *admissible translators*) forming a vector subspace known as the *Cameron-Martin space,* $H(\gamma)$. Then, in fact, γ_h and γ are equivalent, $\gamma \sim \gamma_h$, iff $h \in H(\gamma)$. Indeed, if $\gamma \sim \gamma_h$ fails, then the two measures are mutually singular, $\gamma_h \perp \gamma$ (the Hajek-Feldman Theorem – cf. [Bog1, Th. 2.4.5, 2.7.2]).

Our key inspiration is that, for any non-null measurable subset A of X, the difference set $A-A$ contains a $|.|_H$-open nhd (neighbourhood) of 0 in H, i.e. $(A-A) \cap H$ contains a H-open nhd of 0 – see [Bog1, p. 64]. This flows from the continuity in h of the density of γ_h wrt γ ([Bog1, Cor. 2.4.3]), as given in the *Cameron-Martin-Girsanov formula*:

$$\exp\left(\hat{h}(x) - \frac{1}{2}||\hat{h}||^2_{L^2(\gamma)}\right) \qquad (CM)$$

(where $\hat{h}$ 'Riesz-represents' h, i.e. $x^*(h) = \langle x^*, \hat{h}\rangle$, for $x^* \in X^*$, as in §5). Thus here a modified Steinhaus Theorem holds: the *relative-interior-point theorem.*

In a locally compact topological group, Gaussian measures γ may be defined: see e.g. Heyer [Hey1, 5.2] (in the sense of Parthasarathy; cf. [Hey1, 5.3] for Gaussianity in the sense of Bernstein). See also §9.12. Such a γ may be singular w.r.t. a (left) Haar measure η. Such is the case in the Euclidean case, as above, with the Gaussian having its support on a proper linear subspace H, and in this case $A - A$ with A measurable and non-γ-null will only have non-empty *relative* H-interior (and quasi-invariance only relative to H). We recall two results due to Simmons [Sim] (cf. Mospan [Mos]; for generalizations beyond the locally compact case, using results here, see also the companion paper [BinO7]): (1) a measure μ is singular w.r.t. (left) Haar measure η_G on G if and only if μ is concentrated on a σ-compact subset B such that $B^{-1}B$ has void interior (as in the Euclidean example); (2) μ is absolutely continuous w.r.t. η_G ($\mu \ll \eta_G$) iff the group G has the *Steinhaus property*: for each non-μ-null compact set K,

$$1_G \in \mathrm{int}(K^{-1}K).$$

(This does not preclude having $\mu(K) > 0$ and $\mu(K + h) = 0$ for some K and h.) Other characterizations of (Haar-) absolute continuity and singularity are studied in [LiuR] and [LiuRW], and in the related [Gow1,2,3]; see also [Pro] and [BarFF] on singularity. In certain locally compact groups (e.g. [Hey1, 5.5.7] for the case $G = \mathbb{R}^m \times \mathbb{T}^n$ with $\mathbb{T}$ the unit circle) the condition $\mu \ll \eta_G$ may imply that the support of a Gaussian probability measure μ is G; see, however, [Hey1, 5.5.8] for an example of a Gaussian with full support which is 'Haar-singular'.

We develop an analogue of these relative-interior results for a general Polish group G. This first leads, by analogy with an abstract Wiener space triple [Bog1, 3.9], [Str, 4.2], to the concept of a *Steinhaus triple* (H, G, μ), which we study in §§2-4, demonstrating 'relativized variants' of classical results. In §5 we exhibit a link between the group context and the classical Cameron-Martin theory above by verifying that a divisible abelian group with an $\mathbb{N}$-homogeneous group-norm (below) is in fact a topological vector space. In §6 we extend our usage in §3 of the notions of *subcontinuity* and *selective subcontinuity* of a measure (introduced in [BinO4,7]), and in Theorem 6.1 establish the key property of a *Solecki reference measure.* Then in §7 for a given Polish group G and 'sufficiently subcontinuous' measure μ we construct a corresponding subset $H(\mu)$, which together with G and μ forms a Steinhaus triple (possibly 'selective': see below). We verify that it is an analogue of the Cameron-Martin space when G is a Hilbert space regarded as an additive group. In §8 we examine the extent of the embedded 'subspace' $H(\mu)$. We close with complements in §9.

2 Steinhaus triples: the context

Context. Recall that a *Polish* space X is separable and *topologically complete*, i.e. its topology τ_X may be generated by a complete metric. Throughout the paper G will be a *Polish group* [BecK]: a topological group which is a Polish space. By the Birkhoff-Kakutani Theorem ([Bir], [Kak3]; cf. [DieS, §3.3]) we may equip G with a *left-invariant* metric d_G^L (equivalently, with a (*group-*)*norm* $||g|| := d_G^L(g, 1_G)$, as in [BinO2] – 'pre-norm' in [ArhT]) that generates its topology τ_G. (So $d_G^L(g, g') = ||g^{-1}g'||$ and the corresponding right-invariant metric is $d_G^R(g, g') := ||g'g^{-1}||$.) This metric, which is particularly useful, need not be complete, although $d = d_G^L + d_G^R$ is complete: see [TopH, Th. 2.3.5]. Nonetheless, the group-norm endows G with Fréchet-like features, helpful here. When the norm generating the topology is bi-invariant, *Klee's completeness theorem* [Kle], [DieS, Th. 8.16] asserts that if the topology is completely metrizable, then in fact the norm is itself complete.

We fix a sequence of points $\{g_n\}_{n\in\mathbb{N}}$ dense in G; a sequence $z_n \to 1_G$ will be called *null*, and a null sequence *trivial* if it is ultimately constantly 1_G. For $\delta > 0$, by B_δ^G (resp. B_δ^H) we denote the open ball in G (or H) centered at 1_G of radius δ under d_G^L (or d_H^L), which is *symmetric*; by $\mathcal{B}(G)$ the *Borel* sets; by $\mathcal{K}(G)$ the family of *compact* sets (carrying the Hausdorff metric induced by d_G^L).

Throughout, measure is to mean *Borel measure* – i.e. its domain comprises the Borel sets of the relevant metrizable space [GarP] – and such a measure μ is *Radon* if it is *locally finite* (so that each point has a neighbourhood of finite measure) and the relevant Borel sets are *inner compact regular*, i.e. (μ-)approximable by compacts from within:

$$\mu(B) = \sup\{\mu(K) : K \in \mathcal{K}(G), K \subseteq B\}$$

([Sch] and §9.2); being locally finite on a separable metric space such a measure is σ-finite. A σ-finite measure on a metric space is necessarily *outer regular* ([Kal, Lemma 1.34], cf. [Par, Th. II.1.2] albeit for a probability measure), i.e. approximable by open sets from without, and, when the metric space is completely metrizable, inner regular ([Bog2, II. Th. 7.1.7], [Xia, Th. 1.1.8], cf. [Par, Ths. II.3.1 and 3.2]).

For μ a Radon measure, we write $\mathcal{M}_+(\mu)$ for the non-null sets, and put

$$\mathcal{K}_+(\mu) := \mathcal{K}(G) \cap \mathcal{M}_+(\mu).$$

By $\mathcal{P}(G)$ we denote the family of (Borel) *probability* measures μ on G, i.e. with $\mu(G) = 1$, so these are Radon (as above); by $\mathcal{U}(G)$ the *universally measurable* sets (i.e. measurable with respect to every measure $\mu \in \mathcal{P}(G)$ in the sense of measure completion [Bog2.I, Th. 1.5.6, Prop. 1.5.11 'Lebesgue completion'] – for background and literature, see [BinO7]). We recall that $N \subseteq G$ is (left) *Haar-null* in G if, for some Borel $B \supseteq N$ and $\mu \in \mathcal{P}(G)$, $\mu(gB) = 0$ for all $g \in G$.

Definition. Call (H, G, μ) a *Steinhaus triple* if G is a Polish group with (group-) norm $||.||_G$ (notation as above), μ a Radon measure on G and $1_G \in H \subseteq G$, a *continuously embedded subset* with group-norm $||.||_H$, having the property that for $K \in \mathcal{K}_+(\mu)$ there is in H an (H-) open neighbourhood U of 1_G (i.e. U is open under $||.||_H$) such that

$$U \subseteq K^{-1}K.$$

(So $\mathrm{cl}_G(U)$ is compact in G.) The latter condition links the topological with the algebraic structure; the norm on H introduces a topology on H *finer* than the subspace topology induced by G – cf. [Bog1, Ch. 2], [BerTA], [Gro1,2], [Str, §4.2]. In view of its distinguished status we will refer to H as the *Steinhaus support*. Our aim is to establish structural similarities with the classical Cameron-Martin space $H(\gamma)$ of §1 ([Bog1, §2.4], §3, §8 and §9). Below, τ_G and τ_H will be the topologies of G and H; $\tau_{G|H}$ will be the topology induced on H by τ_G. So

$$\tau_{G|H} \subseteq \tau_H.$$

It will be helpful to bear in mind the following illuminating example.

Cautionary Example. Consider the additive group $G = \mathbb{R}$ with the Euclidean topology and its additive subgroup $H = \mathbb{Q}$ with *discrete* topology. Enumerate $\mathbb{Q}$ as $\{q_n\}_{n\in\mathbb{N}}$ and put

$$\mu := \textstyle\sum_{n=1}^{\infty} 2^{-n}\delta_{q_n},$$

with δ_x the Dirac measure at x. Here $\tau_{G|H} \subseteq \tau_H$. For $K \subseteq \mathbb{R}$ (Euclidean) compact, $\mu(K) > 0$ iff $K \cap \mathbb{Q} \neq \emptyset$, and then $\{0\}$ $(\subseteq K - K)$ is τ_H-open.

The topological link between H and G above is at its neatest and most thematic in norm language. But, as H need not be a subgroup, it would suffice for the continuous embedding to be determined by just a metric on

H, or more generally a choice of refining topology. Furthermore, as in any abstract definition of inner regularity, one is at liberty here to restrict attention to a, possibly countable, subfamily of $\mathcal{K}_+(\mu)$ – see §9.8. Such variants will be referred to below as *selective Steinhaus triples.* See the Remarks after Th. 7.1 and after Prop. 8.3 below.

Remarks. 1. The inclusion above implies that

$$K \cap Ku \neq \emptyset \qquad (u \in U)$$

(indeed, if $u = k^{-1}k'$, then $k' = ku$). In Theorem 3.4 below we strengthen this conclusion to yield the measure-theoretic 'Kemperman property', introduced in [Kem], as in [BinO4]. (One would expect this to imply shift-compactness for 'G-shifts of H', as indeed is so – see Th. 3.8 below.)

2. Note that $W \cap H$ is open in H for W open in G. It follows that if G is not locally compact, then, for $U \subseteq K^{-1}K$ as above, U is *nowhere dense in* G (as $\mathrm{cl}_G U$ is compact, its interior in G must be empty). Theorems 3.1 and 3.2 below expand this remark to stark category/measure dichotomies.

We close this section with an observation which we need several times in the next section.

Lemma 2.1. *For (H, G, μ) a Steinhaus triple, if H is Polish in its own topology, then*

$$\mathcal{B}(H) \subseteq \mathcal{B}(G),$$

i.e. if B is Borel in H, then B is Borel in G. In particular, if $K \subseteq H$ is compact in τ_H, then it is compact in τ_G: $\mathcal{K}(H) \subseteq \mathcal{K}(G)$.

Proof. As H is Polish, B being Borel is a Lusin space (cf. [Sch, Ch. 2], [Rog, 1 § 2.1]), so we may write B as an injective continuous image of the irrationals, $B = f(\mathbb{N}^N)$ say, with $f : \mathbb{N}^{\mathbb{N}} \to H$ continuous (with H under τ_H) [Rog, 1. Cor 2.4.2]. Now the embedding $\iota : H \to G$ is continuous and so $B = \iota \circ f(\mathbb{N}^{\mathbb{N}})$ is an injective continuous image of the irrationals, and so a Borel subset of G [Rog, 1. Th. 3.6.1]. The final assertion is clear (since $K = \iota(K)$ is compact, as $\tau_{G|K} \subseteq \tau_{H|K}$). $\square$

3 Steinhaus triples: the general case

In this section we study general topological properties of Steinhaus triples, foremost among which is *local quasi-invariance* (Theorem 3.5 below), a much weakened version of *relative* quasi-invariance (which we consider separately in the next section), i.e. relative to a subgroup of 'admissible' translators. This is preceded by a technical result (Theorem 3.4) reminiscent of a lemma due to Kemperman [Kem] – cf. [Kuc, Lemma 3.7.2]; this will be revisited in another context in §§6,7. We close the section with Theorem 3.8, deducing a new property of the Cameron-Martin space $H(\gamma)$.

For the sake of clarity, we emphasize that $\mathcal{K}_+(\mu) = \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$: the compactness referred to is thus in the sense of the topology on G. Our first result confirms that the Steinhaus support will always be meagre in our setting:

Theorem 3.1. *For (H, G, μ) a Steinhaus triple with H Polish: if H is a dense subgroup of G, then either*
i) *H is meagre in G (so $\mathrm{int}_G(H) = \emptyset$,), or else*
ii) *G is locally compact and $H = G$.*

Proof. Suppose (i) fails. Choose any $K \in \mathcal{K}_+(\mu)$ and consider any $\delta > 0$ so small that B_δ^H, the H-ball of radius $\delta > 0$ centered at 1_G, satisfies

$$B_\delta^H \cdot B_\delta^H = B_{2\delta}^H \subseteq K^{-1}K.$$

For any countable dense subset D of H

$$H = \bigcup\nolimits_{d \in D} d \cdot B_\delta^H$$

(refer to the metric d_H^L), so $d \cdot B_\delta^H$ is non-meagre in G for some $d \in D$. As G is a topological group, $B_\delta^H = d^{-1}d \cdot B_\delta^H$ is also non-meagre. Now B_δ^H, being open in H, is Borel in G by Lemma 2.1, and so has the Baire property by Nikodym's Theorem [Rog, Part 1 §2.9], and is non-meagre in G. By the Piccard-Pettis Theorem ([Pic], [Pet], cf. [BinO2, Th. 6.5]), applied in G, there is $r > 0$ with

$$B_r^G \subseteq B_\delta^H \cdot B_\delta^H = B_{2\delta}^H \subseteq K^{-1}K.$$

So $B_{2\delta}^H$ for all small enough $\delta > 0$ contains B_r^G for some $r > 0$, and B_r^G has compact closure in G. So $G = H$ (for any $g \in G$ choose $h \in gB_r^G \cap H$; then $g \in hB_r^G \subseteq hB_{2\delta}^H \subseteq H$), the two topologies coincide and G is locally compact. □

Next we turn from category to measure negligibility.

Theorem 3.2. *For (H, G, μ) a Steinhaus triple with H Polish: if H is a dense subgroup of G, then either*
i) *H is μ-null, or else*
ii) *H is locally compact under its own topology, with $\mu_H \ll \eta_H$ for μ_H the restriction of μ to H and η_H a Haar-measure on H.*

Proof. Again suppose (i) fails and again recall that by Lemma 2.1 the open subsets of H have the Baire property as they are Borel in G. Let μ_H denote the restriction of μ to the Borel subsets of H :

$$\mu_H(B) = \mu(B \cap H) \qquad (B \in \mathcal{B}(H)).$$

The resulting measure is still Radon: it is a locally finite (since $\tau_{G|H} \subseteq \tau_H$) Borel measure and H is Polish (cf. §2). So since $\mu_H(H) = \mu(H) > 0$, there is a compact subset $K \subseteq H$ which is μ_H-non-null. This set K is compact also in the sense of G (Lemma 2.1) and μ-non-null. Hence again for some $\delta > 0$

$$B_\delta^H \subseteq K^{-1}K \subseteq H, \tag{KH}$$

the latter inclusion as H is a subgroup. So the topology of H is locally compact (and indeed σ-compact). Hence H supports a left Haar measure η_H and so

$$\mu_H \ll \eta_H,$$

by the Simmons-Mospan theorem ([Sim], [Mos]; [BinO7], §1), since μ_H has the Steinhaus property (§1) on H. So

$$\mu(B) = \mu_H(B \cap H) + \mu(B \backslash H) \qquad (B \in \mathcal{B}(G)),$$

with

$$\mu(B \cap H) = \int_{B \cap H} \frac{d\mu_H}{d\eta_H} d\eta_H(h). \qquad \square$$

Remarks. 1. For H locally compact, as above, the subgroup H of G is capable of being generated by any non-null compact subset K of H. See (KH) above; use a dense set of translates of B_δ^H.
2. For L compact in H, as $L \backslash B_\delta^H$ is closed in H it is also compact in H, and so also in G (via the continuous embedding). Take $L := \mathrm{cl}_H B_\delta^H \subseteq K^{-1}K \subseteq H$; then L is a compact set of H and so of G. So the set $W := G \backslash (L \backslash B_\delta^H)$ is open in G, and so

$$B_\delta^H = L \backslash (L \backslash B_\delta^H) = L \cap W.$$

That is, the topology of H on each subset of the form $L := \mathrm{cl}_H B_\delta^H$ is induced by the topology of G :

$$\tau_{H|L} = \tau_{G|L}.$$

So H is a countable union of compact *subspaces* of G. Compare the Cautionary Example above.
3. Being locally compact, H is topologically complete and so is an absolute $\mathcal{G}_\delta$. But that means only that it is a $\mathcal{G}_\delta$ subset in any space X whereof it is a subspace:

$$\tau_{X|H} = \tau_H.$$

A complementary result follows, in which we need to assume that, for τ_G-compact subsets of H, the mapping $m_K(h) := \mu(Kh)$ is continuous on H at 1_G relative to $\tau_{G|H}$, the topology induced by G. (This is a relativized version of the global concept of *mobility* studied by A. van Rooij and his collaborators – see e.g. [LiuR].) The proof below is a relativized version of one in [Gow1]; we give it here as it is short and thematic for our development.

Theorem 3.2′ (cf. [Gow1], [Hey1, Lemma 6.3.4]). *For (H, G, μ) a Steinhaus triple with H Polish and dense in G, and G not locally compact*: *if $h \mapsto \mu(Kh)$ is $\tau_{G|H}$-continuous at 1_G on H for each $K \in \mathcal{K}_+(\mu)$ lying in H, then $\mu(H) = 0$.*

Proof. Suppose otherwise. Then, referring as in Th. 3.2 to the restriction $\mu_H(B) = \mu(B)$ for $B \in \mathcal{B}(H)$, there is $K \subseteq H$ compact in H (so also compact in G) with $\mu(K) > 0$. Consider any $\delta > 0$ with $0 < \delta < \mu(K)$. Put $\varepsilon := \delta/3$. Choose open U in G with $K \subseteq U$ such that $\mu(U) < \mu(K) + \varepsilon$, and $\eta > 0$ so that

$$|\mu(Kh) - \mu(K)| \leq \varepsilon$$

for $h \in B_\eta^G \cap H$. W.l.o.g. $KB_\eta^G \subseteq U$, whence $\mu(Kh \backslash K) \leq \mu(U \backslash K) \leq \varepsilon$, for $h \in B_\eta^G \cap H$, and likewise $\mu(K \backslash Kh) \leq 2\varepsilon$, because

$$\mu(K) - \varepsilon \leq \mu(Kh) \leq \mu(Kh \cap K) + \mu(U \backslash K).$$

Combining yields

$$\mu(Kh \triangle K) \leq 3\varepsilon = \delta.$$

So, since $\delta < \mu(K)$,

$$B_\eta^G \cap H \subseteq \{h \in H : \mu(Kh \triangle K) \leq \delta\} \subseteq \{h \in H : Kh \cap K \neq \emptyset\} \subseteq K^{-1}K :$$

$$H \cap B^G_\eta \subseteq K^{-1}K.$$

So

$$\mathrm{cl}_G(B^G_\delta) = \mathrm{cl}_G(H \cap B^G_\delta) \subseteq K^{-1}K.$$

Hence G is locally compact, a contradiction. □

Theorem 3.3. *For (H, G, μ) a Steinhaus triple, with H Polish, if H is a dense proper subgroup of G, then H is generically left Haar-null in G – left Haar-null for quasi all $\mu' \in \mathcal{P}(G)$, in the sense of the Lévy metric on $\mathcal{P}(G)$. In particular, this is so for the Cameron-Martin space $H(\gamma)$.*

Proof. This follows from a result of Dodos [Dod, Cor. 9] since H is an analytic subgroup (cf. Lemma 2.1) with empty interior (Th. 3.1, as $H \neq G$). □

We include here a similar result which is thematic.

Theorem 3.3′ (Smallness). *For X a Fréchet space carrying a Radon Gaussian measure γ, $H(\gamma)$ is (generically) left Haar-null – left Haar-null for quasi all $\mu \in \mathcal{P}(X)$, in the sense of the Lévy metric on $\mathcal{P}(X)$.*

Proof. For γ a Radon probability measure, $H := H(\gamma)$ is a separable Hilbert space ([Bog1, 3.2.7], cf. [Bog1, p. 62]). So it is complete under its own norm, so a Polish space; it is continuously embedded in X [Bog1, 2.4.6] so, as a subset of X, it is analytic as in Lemma 2.1 (here: $\tau_{X|H} \subseteq \tau_H$). As H has empty interior in X (Th. 3.1), again by the result of Dodos [Dod, Cor. 9], H is generically left-Haar null. □

Remark. In the setting above, $H := H(\gamma)$ is in fact a σ-*compact* subset of X: by [Bog1, 2.4.6], the H-closed *unit ball* U_H is weakly closed in X and, being convex (by virtue of its norm), it is closed in X [Rud2, 3.12], cf. [Con, §5.12]; but, as in the next theorem (Th. 3.4), it is a subset of some compact set of the form $K - K$. So U_H is itself compact in X – see [Bog1, 3.2.4].

Of course, if X is an infinite-dimensional space Hilbert space and H is a σ-compact subset of X, then, by Baire's theorem, H must have empty interior in X.

Theorem 3.4. *For (H, G, μ) a Steinhaus triple and $K \in \mathcal{K}_+(\mu)$, there are $\varepsilon, \delta > 0$ such that*

$$\mu(K \cap Kh) \geq \delta \text{ for all } h \in H \text{ with } ||h||_H \leq \varepsilon.$$

In particular,

$$\mu_-^H(K) := \sup_{\varepsilon>0} \inf\{\mu(Kh) : h \in B_\varepsilon^H\} \geq \delta,$$

so that for any null sequence $\mathbf{t} = \{t_n\}$ *in* H *(i.e. with* $t_n \to_H 1_G$*),*

$$\mu_-^{\mathbf{t}}(K) := \liminf_{n\to\infty} \mu(Kt_n) \geq \delta;$$

furthermore, for some $r > 0$*,*

$$K \cap Kh \in \mathcal{M}_+(\mu) \qquad (h \in B_r^H): \qquad B_r^H \subseteq K^{-1}K.$$

Proof. Suppose otherwise. Then for some $K \in \mathcal{K}_+(\mu)$ and for each pair $\varepsilon, \delta > 0$ there is $h \in H$ with $||h||_H < \varepsilon$ and

$$\mu(K \cap Kh) < \delta.$$

So in H there is a sequence $t_n \to_H 1_G$ with

$$\mu(K \cap Kt_n) < 2^{-n-1}\mu(K)$$

for each $n \in \mathbb{N}$. Take

$$M := K \cap \bigcup\nolimits_{n\in\mathbb{N}} Kt_n.$$

Then

$$\mu(M) < \mu(K)/2.$$

So we may choose a compact μ-non-null $K_0 \subseteq K\backslash M$; then, since (H, G, μ) is a Steinhaus triple, there is in H a non-empty open nhd V of 1_G with

$$V \subseteq K_0^{-1}K_0.$$

Now $t_m \in V$ for all large m. Fix such an m; then

$$K_0 \cap K_0 t_m \neq \emptyset.$$

Consider $k_0, k_1 \in K_0$ with $k_0 = k_1 t_m \in K_0 \cap K_0 t_m \subseteq K \cap \bigcup_n Kt_n = M$; as K_0 is disjoint from M, this is a contradiction. $\square$

An immediate corollary is

Theorem 3.5 (Local quasi-invariance). *For (H, G, μ) a Steinhaus triple and $B \in \mathcal{B}(G)$: if $\mu(B) > 0$, then there exists $\delta > 0$ so that $\mu(Bh) > 0$ for all $h \in H$ with $||h||_H \leq \delta$.*

Proof. This follows from Th. 3.4 since there is compact $K \subseteq B$ with $\mu(K) > 0$. Then for all sufficiently small $h \in H$ $\mu(Bh) \geq \mu(Kh) > 0$. □

Corollary 3.1. *For each null sequence $t_n \to 1_G$ in H and μ-non-null K,*

$$0 < \mu_{-}(K) \leq \liminf \mu(t_n K) \leq \limsup \mu(t_n K) \leq \mu(K),$$

and so, for μ-non-null compact $L \subseteq K$,

$$0 < \mu_{-}(L) \leq \mu_{-}(K) \text{ and } \mu(L) \leq \mu(K).$$

Proof. Writing $\mu^{\delta}(K) := \inf\{\mu(Kh) : h \in B_{\delta}^{H}\}$, for any $\delta > 0$ and all large enough n

$$\mu^{\delta}(K) \leq \mu(t_n K) \leq \mu^{\delta}(K) + \delta,$$

yielding the lower bound when $\delta \downarrow 0$. Also for $\delta > 0$, there is open $U \supseteq K$ with $\mu(U \backslash K) \leq \delta$, and so, as K is compact, for all large enough n

$$\mu(t_n K) \leq \mu(K) + \delta.$$

For each $\delta > 0$ choose $t_{\delta} \in B_{\delta}^{H}$ with $\mu^{\delta}(K) \leq \mu(t_{\delta} K) \leq \mu^{\delta}(K) + \delta$. Then, for non-null $L \subseteq K$, since $t_{\delta} L \subseteq t_{\delta} K$, by the earlier proved assertions,

$$\mu_{-}(L) \leq \liminf \mu(t_{\delta} L) \leq \liminf \mu(t_{\delta} K) \leq \liminf[\mu^{\delta}(K) + \delta] = \mu_{-}(K),$$

and

$$\limsup \mu(t_{\delta} L) \leq \mu(L) \leq \mu(K). \qquad \square$$

Remarks. 1. Above, if $\mu(t_n K) \to \mu_0 \geq \mu_{-}(K)$, then

$$\mu(\textstyle\bigcap_{n \in \mathbb{N}} (K \backslash t_n K)) = \mu(K) - \mu_0 \leq \mu(K) - \mu_{-}(K).$$

2. Evidently, for disjoint non-null compact K, L

$$\mu(K) + \mu(L) = \mu(K \cup L) \geq \mu_{-}(K \cup L) \geq \mu_{-}(K) + \mu_{-}(L).$$

If one of these is sharp, one may imagine passsing through a subsequence K_n with $\mu(K_n) > \mu_{-}(K_n) > 0$.

There is no Steinhaus-like assumption on the measure μ in the following result, which, standing in apposition to Th. 3.4, is a kind of converse.

Proposition 3.1 ([BinO7, L. 1]). *For $H \subseteq G$ continuously embedded in G, $\mu \in \mathcal{P}(G)$ and $K \in \mathcal{K}_+(\mu)$: if $\mu_-^H(K) > 0$, then there are $\delta, \Delta > 0$ with*

$$\Delta/4 \leq \mu(K \cap Kt) \qquad (t \in B_\delta^H),$$

so that

$$K \cap Kt \in \mathcal{M}_+(\mu) \qquad (t \in B_\delta^H). \tag{$*$}$$

In particular,

$$K \cap Kt \neq \emptyset \qquad (t \in B_\delta^H),$$

or, equivalently,

$$B_\delta^H \subseteq K^{-1}K, \tag{$**$}$$

so that B_δ^H has compact closure under τ_G.

Proof. Put $H_t := K \cap Kt \subseteq K$. Take $\Delta := \mu_-^H(K) > 0$. Then for any small enough $\delta > 0$, $\mu(Kt) > \Delta/2$ for $t \in B_\delta^H$. Fix such a $\delta > 0$.

By outer regularity of μ, choose U open with $K \subseteq U$ and $\mu(U) < \mu(K) + \Delta/4$. By upper semicontinuity of $t \mapsto Kt$, w.l.o.g. $KB_\delta \subseteq U$. For $t \in B_\delta^H$, by finite additivity of μ, since $\Delta/2 < \mu(Kt)$

$$\begin{aligned} \Delta/2 + \mu(K) - \mu(H_t) &\leq \mu(K) + \mu(Kt) - \mu(H_t) = \mu(K \cup Kt) \\ &\leq \mu(U) \leq \mu(K) + \Delta/4. \end{aligned}$$

Comparing extreme ends of this chain of inequalities gives

$$0 < \Delta/4 \leq \mu(H_t) \qquad (t \in B_\delta^H).$$

For $t \in B_\delta^H$, as $K \cap Kt \in \mathcal{K}_+(\mu)$, take $s \in K \cap Kt \neq \emptyset$; then $s = at$ for some $a \in K$, so $t = a^{-1}s \in K^{-1}K$. Conversely, $t \in B_\delta^H \subseteq K^{-1}K$ yields $t = a^{-1}a'$ for some $a, a' \in K$; then $a' = at \in K \cap Kt$. □

Löwner showed in 1939 that there exists no (σ-finite) translation-invariant measure on an infinite-dimensional Hilbert space ([Loe, §1], [Neu2]). This is contained in the result below: with G a Hilbert space regarded as an additive group and μ Radon, if μ is translation-invariant, then $\mu(K) = \mu_-^G(K) > 0$ for some compact K, so G is locally compact and so finite-dimensional.

Corollary 3.2 (cf. [Gow1]). *If $\mu_-^G(K) > 0$ for some $K \in \mathcal{K}_+(\mu)$, then G is locally compact.*

Proof. If $\mu_-(K) > 0$, then there is $\delta > 0$ such that $\mu(tK) > \mu_-(K)/2$ for all $t \in B_\delta = B_\delta^G$; so for $\Delta := \mu_-(K)/2$

$$B_\delta \subseteq B_\delta^\Delta = \{z \in B_\delta : \mu(Kz) > \Delta\} \subseteq K^{-1}K,$$

and so B_δ has compact closure. □

Theorem 3.6. *For (H, G, μ) a Steinhaus triple and $K \in \mathcal{K}_+(\mu)$, the set*

$$\mathcal{O}(K) := \{h \in H : \mu(Kh) > 0\}$$

is H-open, and so μ is continuous on a dense $\mathcal{G}_\delta$ of $\mathcal{O}(K)$ (so off a H-meagre set), i.e.

$$\mu(Kh) = \mu_-^H(Kh) > 0 \qquad \text{(quasi all } h \in \mathcal{O}(K)).$$

Conversely, for H, G topological groups with H a continuously embedded subgroup of G and $\mu \in \mathcal{P}(G)$: if $\mathcal{O}(K)$ is open for $K \in \mathcal{K}_+(\mu)$ and the above relative continuity property of μ holds on a dense $\mathcal{G}_\delta$ subset of $\mathcal{O}(K)$ in H, then (H, G, μ) is a Steinhaus triple.

Proof. The first assertion follows from Th. 3.5 on local quasi-invariance applied to Kh with $h \in \mathcal{O}(K)$. Since the map $g \mapsto \mu(Kg)$ is upper semi-continuous (see. e.g. [BinO7, Prop. 1], [Hey1, 1.2.8]), the second assertion follows from the first by the theorem of Fort [For, R1] (cf. [Xia, Appendix I, Lemma I.2.2]) that an upper semi-continuous map is continuous on a H-dense $\mathcal{G}_\delta$ in $\mathcal{O}(K)$.

As for the converse, for $K \in \mathcal{K}_+(\mu)$, since $1_G \in \mathcal{O}(K)$ there is $h \in H$ with $\mu(Kh) = \mu_-^H(Kh) > 0$. It now follows by Th. 3.4 that $B_r^H \subseteq (Kh)^{-1}Kh$ for some $r > 0$, and so $1_G \in hB_r^H h^{-1} \subseteq K^{-1}K$, i.e. 1_G is an interior point of $K^{-1}K$ (since $hB_r^H h^{-1}$ is open, as H is a topological group). □

As a corollary of Th. 3.4, we now obtain a result concerning embeddability into non-negligible sets (here the non-null measurable sets) of some translated subsequence of a given null sequence. This property, first used implicitly by Banach [Ban1,2], has been studied in various general contexts by many authors, most recently under the term ‘shift-compactness’ – see e.g. [Ost1]. The new context of a Steinhaus triple is notable in limiting the null sequences to the distinguished subset. Here the statement calls for the passage from a null sequence in H to its inverse sequence; this inversion is of course unnecessary if $H^{-1} = H$, e.g. if H is a subgroup of G, as will be the case in Theorem 3.8 below. (The group-theoretic approach to shift-compactness is that of a group action, here of translation in G – see [MilO]; for applications see [Ost2].)

Theorem 3.7 (Shift-compactness Theorem for Steinhaus triples). *For (H, G, μ) a Steinhaus triple,* $\mathbf{h}$ *a null sequence in H, and $E \in \mathcal{M}_+(\mu)$: for μ-almost all $s \in E$ there exists an infinite $\mathbb{M}_s \subseteq \mathbb{N}$ with*

$$\{sh_m^{-1} : m \in \mathbb{M}_s\} \subseteq E.$$

Proof. Fix a compact $K_0 \subseteq E$ with $\mu(K_0) > 0$. Choose inductively a sequence $m(n) \in \mathbb{N}$ and decreasing compact sets $K_n \subseteq K_0 \subseteq E$ with $\mu(K_n) > 0$ such that

$$\mu(K_n \cap K_n h_{m(n)}) > 0.$$

To check the inductive step, suppose K_n already defined. As $\mu(K_n) > 0$, by Th. 3.4 there are $\delta, \varepsilon > 0$ such that

$$\mu(K_n \cap K_n h) \geq \delta \text{ for all } h \in H \text{ with } ||h||_H \leq \varepsilon.$$

So there is $m(n) > n$ with $\mu(K_n \cap K_n h_{m(n)}) > 0$. Putting $K_{n+1} := K_n \cap K_n h_{m(n)} \subseteq K_n$ completes the inductive step, and so the induction.

By compactness, select s with

$$s \in \bigcap\nolimits_{m \in \mathbb{N}} K_m \subseteq K_{n+1} = K_n \cap K_n h_{m(n)} \qquad (n \in \mathbb{N}).$$

Choosing $k_n \in K_n \subseteq K$ with $s := k_n h_{m(n)}$ gives $s \in K_0 \subseteq E$, and

$$sh_{m(n)}^{-1} = k_n \in K_n \subseteq K_0 \subseteq E.$$

Finally take $\mathbb{M} := \{m(n) : n \in \mathbb{N}\}$.

As for the final assertion, recalling from §2 that $\mathcal{U}(G)$ denotes the universally measurable sets, define

$$F(H) := \bigcap\nolimits_{n \in \mathbb{N}} \bigcup\nolimits_{m > n} H \cap H h_m \qquad (H \in \mathcal{U}(G)).$$

Then $F : \mathcal{U}(G) \to \mathcal{U}(G)$ and F is monotone ($F(S) \subseteq F(T)$ for $S \subseteq T$); moreover, $s \in F(H)$ iff $s \in H$ and $sh_m^{-1} \in H$ for infinitely many m. It suffices to show that $E_0 := E \backslash F(E)$ is μ-null (cf. the Generic Completeness Principle [BinO1, Th. 3.4]). Suppose otherwise. Then, as $\mu(E_0) > 0$, there exists a compact $K_0 \subseteq E_0$ with $\mu(K_0) > 0$. But then, as in the construction above, $\emptyset \neq F(K_0) \cap K_0 \subseteq F(E) \cap E_0$, contradicting $F(E) \cap E_0 = \emptyset$. □

Corollary 3.3. *If the subsequence embedding property of Theorem 3.7 holds for all the null sequences in a set H which is continuously embedded in G for all $E \in \mathcal{K}_+(\mu)$, then (H, G, μ) is a Steinhaus triple. In particular, for any $E \in \mathcal{M}_+(\mu)$ the set $E^{-1}E$ has non-empty H-interior.*

Proof. If in H there is no open subset U with $U \subseteq K^{-1}K$, then there exists $h_n \in H$ with $h_n \notin B_{1/n}^H \backslash (K^{-1}K)$. Then there is $s \in K$ and an infinite $\mathbb{M}_s \subseteq \mathbb{N}$ with

$$\{sh_m^{-1} : m \in \mathbb{M}_s\} \subseteq K.$$

So for any $m \in \mathbb{M}_s$, $h_m s^{-1} \in K^{-1}$, i.e. $h_m \in K^{-1}K$, a contradiction. □

Another immediate corollary is the following result, which was actually our point of departure.

Theorem 3.8 (Shift-compactness Theorem for the Cameron-Martin Space). *For X a Fréchet space carrying a Radon Gaussian measure γ with $X^* \subseteq L^2(\gamma)$, and $H(\gamma)$ the Cameron-Martin space: if $\mathbf{h}$ is null in $H(\gamma)$, and $E \in \mathcal{M}_+(\gamma)$, then for γ-almost all $s \in E$ there exists an infinite $\mathbb{M}_s \subseteq \mathbb{N}$ with*

$$\{s + h_m : m \in \mathbb{M}_s\} \subseteq E.$$

Proof. Regarding X as an additive group, $(H(\gamma), X, \gamma)$ is a Steinhaus triple, by [Bog1, p. 64]. As $H(\gamma)$ is a subspace of X, $(-h_n)$ is also a null sequence in $H(\gamma)$; by Theorem 3.7, for γ-almost all $s \in E$ there is $\mathbb{M}_s \subseteq \mathbb{N}$ with

$$\{s - (-h_m) : m \in \mathbb{M}_s\} \subseteq E. \qquad \square$$

4 Steinhaus triples: the quasi-invariant case

We begin with the definition of measure 'relative quasi-invariance' promised in §3. Classical results on this topic are given in the setting of topological vector spaces – see e.g. [GikS, Ch. VII], [Sko], [Yam2] – with the exception of [Xia] which develops the associated harmonic analysis in its group setting. We adopt a similar approach here in order to pursue some parallels with Cameron-Martin theory.

4.1 Relative quasi-invariance

Definition. Say that μ is *relatively quasi-invariant w.r.t* H, or just H*-quasi-invariant,* if $\mu(hB) = 0$ for all μ-null Borel $B \in \mathcal{B}(G)$ (equivalently, μ-null compact B) and all $h \in H$.

Recall that $\mathrm{supp}_G(\mu)$ denotes the *topological support*, which is the smallest closed set of full μ-measure; for μ Radon, such a smallest closed set is guaranteed to exist [Bog2, Prop. 7.2.9].

Proposition 4.1 (cf. [Bog1, 3.6.1])*. For* (H, G, μ) *a Steinhaus triple with* H *a subgroup and* $\mu \in \mathcal{P}(G)$*, a* H*-quasi-invariant (Radon) measure: if* $\mu(\mathrm{cl}_G H) = 1$*, then*

$$S_\mu := \mathrm{supp}_G(\mu) = \mathrm{cl}_G H.$$

In particular, this is so for H *the Cameron-Martin space* $H(\gamma)$*.*

Proof. Let $L := \mathrm{cl}_G H$; then $L \subseteq S_\mu$. If the inclusion were proper: take $x \in L \backslash S_\mu$. There is V open in G with $x \in V$ and $\mu(V) = 0$. But, as $x \in L$, there is $h_0 \in V \cap H$, and so $1_G \in W := h_0^{-1} V$ with $\mu(W) = 0$. W.l.o.g. $W^{-1} = W$ (otherwise pass to $W \cap W^{-1}$, which contains 1_G). So also $\mu(hW) = 0$ for $h \in H$ (by H-quasi-invariance). Then, by the definition of the support,

$$HW = \bigcup\nolimits_{h \in H} (hW) \subseteq X \backslash S_\mu,$$

and so $\mu(HW) = 0$. But $\mathrm{cl}_G H \subseteq HW$, for if the point $x \in \mathrm{cl}_G H$, then its nhd xW meets H, in h say; then $x \in hW^{-1} = hW$ (since $xw = h$ implies $x = hw^{-1}$). So $\mu(\mathrm{cl}_G H) = 0$, contradicting the fact that $\mu(\mathrm{cl}_G H) = 1$. $\square$

Definition. Following [Bog1, 3.6.2], say that for a Steinhaus triple (H, G, μ) the measure μ is *non-degenerate* iff $S_\mu = \mathrm{cl}_G H$.

We close this subsection by tracing a measure-to-category dependence.

Proposition 4.2 (From measure to category: nullity to empty interior). *For (H, G, μ) a Steinhaus triple with H a subgroup and $\mu \in \mathcal{P}(G)$ an H-quasi-invariant non-degenerate (Radon) measure: if $\mu(H) = 0$, then*

$$\mathrm{int}_G(H) = \emptyset.$$

In particular, this is so for H the Cameron-Martin space $H(\gamma)$.

Proof. Suppose not. Then w.l.o.g. $1_G \in W :=\mathrm{int}_G(H) \subseteq H$ (as H is a subgroup and so $Ww^{-1} \subseteq H$ for $w \in W$). Also w.lo.g. $W = W^{-1}$ (otherwise pass to the non-empty open subset $W^{-1} \cap W \subseteq H$). But $\mu(H) = 0$, so $\mu(W) = 0$, and so also $\mu(hW) = 0$ for $h \in H$ (by H-quasi-invariance). Then, by the definition of the support,

$$HW = \bigcup\nolimits_{h \in H} hW \subseteq X \backslash S_\mu.$$

So $\mu(HW) = 0$, and the rest of the proof is as in Prop. 4.1 using $S_\mu = \mathrm{cl}_G H$.
□

4.2 Admissible translators for μ-quasi-invariance

We close with a study of the algebraic structure of admissible translators by considering the natural candidates for the Steinhaus support of a (Radon) measure and a corresponding natural complement (inspired by the Hajek-Feldman Dichotomy Theorem – cf. [Bog1, Th. 2.4.5, 2.7.2]). The results here (in particular Prop.4.5) will be used in §8. In what follows the use of Q ('q for quasi-invariance') is justified in Prop. 4.4 below; write $\mu_g(B) := \mu(Bg)$ for $B \in \mathcal{B}(G)$, and put

$$\begin{aligned} Q_R &= Q_R(\mu) := \{g \in G : (\forall K \in \mathcal{K}_+(\mu))\ \mu(Kg) > 0\}, \\ Q &= Q(\mu) := \{g \in G : (\forall K \in \mathcal{K}_+(\mu))[\mu(Kg) > 0\ \&\ \mu(Kg^{-1}) > 0]\}, \\ \mathcal{N} &= \mathcal{N}(\mu) := \{g \in G : \mu_g \perp \mu\}, \qquad G_0 = G_0(D) := \bigcup\nolimits_{d \in D} d\mathcal{N}(\mu), \end{aligned}$$

where $D := \{g_n : n \in \mathbb{N}\}$ is a dense subset of G. Evidently

$$\mathcal{N}(\mu) = \bigcap\nolimits_{K \in \mathcal{K}_+(\mu)} \{g : \mu(Kg) = 0\} = \bigcap\nolimits_{n \in \mathbb{N}} \bigcap\nolimits_{K \in \mathcal{K}_+(\mu)} \{g : \mu(Kg) < 1/n\}.$$

The definitions imply that

$$Q \subseteq Q_R \subseteq G \backslash \mathcal{N}(\mu).$$

Lemma 4.1. *Q is a subgroup of G.*

Proof. First, $g \in Q$ iff $g^{-1} \in Q$. Next take $x, y \in Q$ and $K \in \mathcal{K}_+(\mu)$. As $Kx \in \mathcal{K}_+(\mu)$ and $y \in Q$, $\mu(Kxy) = \mu((Kx)y) > 0$. $\square$

Proposition 4.3. *Q is the largest subgroup of admissible translators for μ-right quasi-invariance:*

$$\mu_g \sim \mu \text{ for } g \in Q.$$

In particular, if Q_R is a subgroup, then $Q = Q_R$.

Proof. For $K \in \mathcal{K}(G)$: if $\mu(K) = 0$ and $g \in Q$, then $\mu(Kg) = 0$. For suppose otherwise. Then $\mu(Kg) > 0$, and then also $\mu((Kg)g^{-1}) > 0$, as $g^{-1} \in Q$, i.e. $\mu(K) > 0$, a contradiction. Evidently, if g admits right quasi-invariance, then $\mu(Kg) > 0$ for $K \in \mathcal{K}_+(\mu)$; so if g lies in a subgroup admitting right quasi-invariance, then also $\mu(Kg^{-1}) > 0$, and so $g \in Q$. $\square$

Proposition 4.4 (Subgroups). *For G abelian and symmetric $\mu \in \mathcal{P}(G)$, Q_R is a subgroup of G. In particular, for G a Hilbert space as in §1, equipped with a symmetric Gaussian measure $\mu = \gamma$, the Cameron-Martin space $H(\gamma)$ is precisely of the form Q_R.*

Proof. For μ symmetric, and $K \in \mathcal{K}(G)$: if $\mu(K) > 0$, then $\mu(K^{-1}) > 0$; so, for $x \in Q_R$, $\mu(K^{-1}x) > 0$. But, as G is abelian, by symmetry

$$\mu(Kx^{-1}) = \mu(xK^{-1}) = \mu(K^{-1}x) > 0.$$

So if $x \in Q_R$, then $x^{-1} \in Q_R$, i.e. $Q_R = Q$, and, by Lemma 4.1, Q_R is a group.

In particular, for any (symmetric) Gaussian $\mu = \gamma$ with domain a Hilbert space G, as in §1, the Cameron-Martin space $H(\gamma)$ coincides with $Q_R(\gamma)$. Indeed, the Hilbert space G, regarded as an additive group, is abelian, and $h \in H(\gamma)$ holds iff μ_h is equivalent to μ. This may be re-stated as follows: $h \in H(\gamma)$ holds iff for all $K \in \mathcal{K}(G)$: $\mu(K) > 0$ iff $\gamma_h(K) > 0$. $\square$

Lemma 4.2. $Q_R\mathcal{N}(\mu) \subseteq \mathcal{N}(\mu)$ and $Q\mathcal{N}(\mu) = \mathcal{N}(\mu)$.

Proof. For $h \in Q$, observe that $\mathcal{K}_+(\mu)h = \mathcal{K}_+(\mu)$: indeed, if $h \in Q$, then $\mu(Kh) > 0$ for all $K \in \mathcal{K}_+(\mu)$, so that $\mathcal{K}_+(\mu)h \subseteq \mathcal{K}_+(\mu)$. Further, for any compact $K \in \mathcal{K}_+(\mu)$, as $h^{-1} \in Q$ so that $\mu(Kh^{-1}) > 0$, $K = Kh^{-1}h \in \mathcal{K}_+(\mu)h$.

It now follows that if $\mu(Kg) = 0$ for all $K \in \mathcal{K}_+(\mu)$, then $\mu(Khg) = 0$ for $h \in Q$ and all $K \in \mathcal{K}_+(\mu)$, i.e. $hg \in \mathcal{N}(\mu)$, i.e. $Q\mathcal{N}(\mu) \subseteq \mathcal{N}(\mu)$. But if $g \in \mathcal{N}(\mu)$ and $h \in Q$, then likewise $h^{-1}g \in \mathcal{N}(\mu)$, as $h^{-1} \in Q$, so $g = h(h^{-1}g) \in Q\mathcal{N}(\mu)$. □

Proposition 4.5. *$\mathcal{N}(\mu)$ is Borel.*

We give two proofs below, the second of which was kindly suggested by the Referee.

First Proof. The sets $\{g : \mu(Kg) < 1/n\}$ are open as $x \mapsto \mu(Kx)$ is upper semicontinuous for K compact. Indeed, the set

$$\{(g, K) : \mu(Kg) < 1/n\}$$

is open in the product space $G \times \mathcal{K}(G)$: for open $U \supseteq K$ with $\mu(U) < 1/2n$, choose $\delta > 0$ with $KgB_{2\delta} \subseteq U$. Then for compact $H \subseteq KB_\delta$ and $h \in gB_\delta$,

$$Hh \subseteq KgB_{2\delta} \subseteq U.$$

By inner regularity we may assume that μ is concentrated on a σ-compact set, say on $\bigcup_n K_n$ with K_n an ascending sequence of compact sets. Then $\{K \subseteq K_m : \mu(K) = 0\} = \bigcap_{n\in\mathbb{N}}\{K \subseteq K_m : \mu(K) < 1/n\}$ is $\mathcal{G}_\delta$ in $\mathcal{K}(K_m)$, and its complement an $\mathcal{F}_\sigma$ in $\mathcal{K}(G)$, as $\mathcal{K}(K_m)$ is compact. Consequently,

$$\{(g, K) : K \in \mathcal{K}_+(\mu), \mu(Kg) \geq 1/n\} = \bigcup_m \{(g, K) : K_m \cap K \in \mathcal{K}_+(\mu), \mu(Kg) \geq 1/n\}$$

$$= \{(g, K) : \mu(Kg) \geq 1/n\} \cap \bigcup_m \{(g, K) : K_m \cap K \in \mathcal{K}_+(\mu)\} \in \mathcal{F}_\sigma(G \times \mathcal{K}(G)).$$

So

$$G\backslash\mathcal{N}(\mu) = \mathrm{proj}_G \bigcup_{m,n} \{(g, K) : K_m \cap K \in \mathcal{K}_+(\mu), \mu(Kg) \geq 1/n\}.$$

But the vertical sections $\{g\} \times \bigcup_m \{K : K_m \cap K \in \mathcal{K}_+(\mu), \mu(Kg) \geq 1/n\}$ are σ-compact. So the projection is Borel, by the Arsenin-Kunugui theorem ([Rog, Th. 1.4.3, Th. 5.9.1], cf. [Kec, Th. 18.18]). □

Second Proof. Assume w.l.o.g. that $\mu \in \mathcal{P}(G)$ so that also $\mu_g \in \mathcal{P}(G)$ for $g \in G$. (Otherwise, since μ is σ-finite (§2), we may replace μ by an equivalent probability measure μ'; then $\mu_g \perp \mu$ iff $\mu'_g \perp \mu'$.) Since G is separable, we may choose a (dense) sequence of continuous real-valued functions $\{f_n\}$ with $||f_n||_\infty \leq 1$ so that, with $||.||_{TV}$ denoting the total variation distance,

$$||\mu - \mu_g||_{TV} = \sup_n \int f_n d(\mu - \mu_g) \quad (g \in G);$$

cf. [Con, App. C Cor. C.14], [Yos, I.3 Cor.]. Take $\{K_m\}_m$ an increasing sequence of compact sets with μ concentrated on on its union; then f_{nm}, where

$$f_{nm}(g) := (f_n * 1_{K_m^{-1}})(g^{-1})$$

(with $*$ denoting convolution), is continuous, cf. [HewR, 20.16] or [Rud1, §1.1.5-1.1.6]. But

$$f_{nm}(g) = \int_{K_m} f_n(yg^{-1}) d\mu(y) \to \int f_n(x) d\mu_g(x) \qquad (\text{as } m \to \infty).$$

So

$$||\mu - \mu_g||_{TV} = \sup_n \lim_m \int_{K_m} [f_n(y) - f_n(yg^{-1})] d\mu(y),$$

and so $f(g) := ||\mu - \mu_g||_{TV}$ $(g \in G)$ is Borel measurable. But, cf. [Bog1, Problem 2.21], $\mathcal{N}(\mu) = f^{-1}\{2\}$, which is thus Borel. □

5 Groups versus vector spaces

Here we link our new results, in a group context, to classical Cameron-Martin theory, in a topological vector space context. There is some similarity here to material in [Hey1, §3.4] on (homomorphic) embeddings of $\mathbb{Q}$ (rational embeddability) and of $\mathbb{R}$ (the more exacting, continuous embeddability) in the space of probability measures; for later developments see [Hey2], [HeyP]. The latter are related to divisibility properties of groups (and of the convolution semigroups of measures). Recall that a group G is (infinitely, or $\mathbb{N}$-) *divisible* [HewR, A5] if for each $n \in \mathbb{N}$ every element $g \in G$ has an n-th root $h \in G$, i.e. with $h^n = g$ (for their structure theory in the abelian case, see e.g. [HewR, A14], [Fuc], or [Kap]).

The refinement norm used to define the classical Cameron-Martin space within a topological vector space X depends ultimately on the embedding of the continuous linear functionals, X^*, in $L^2(X,\gamma)$ and on the Riesz Representation theorem for Hilbert spaces. In common with the additive-subgroup literature of topological vector spaces (for the corresponding Pontryagin-van Kampen duality theory, see e.g. Gel′fand-Vilenkin [GelV], [Bana], [Mor], and [Xia]), we consider the natural group analogue here to be the embedding of continuous real-valued additive maps on a metrizable abelian group X into $L^2(X,\mu)$ for $\mu \in \mathcal{P}(X)$, with mean ('averaging map')

$$a_\mu : x^* \mapsto \mu(x^*) := \int_X x^*(x)d\mu(x) \qquad (x^* \in X^*),$$

with X^* the continuous additive maps on X into $\mathbb{R}$. One may then, as in the classical setting ([Bog1, 2.2]), define a covariance operator by

$$R(x^*)(y^*) := \int_X [x^*(x) - \mu(x^*)][y^*(x) - \mu(y^*)]d\mu(x) \qquad (x^*, y^* \in X^*).$$

As there, for $h \in X$ define the *Cameron-Martin group-norm*

$$|h|_H := \sup\{x^*(h) : x^* \in X^*, R(x^*, x^*) \leq 1\},$$

$$H := \{h \in X : |h|_H < \infty\}.$$

It may now readily be checked that H is a subgroup, and that $|h|_H$ is a group-norm on H. (For any $h \in X$, with $h \neq 1_X$, use a standard extension theorem, e.g. as in [HewR, A.7], to extend the partial homomorphism $h^s \mapsto s$ (for $s \in \mathbb{Z}$) to a full homomorphism x_h^*, say of unit variance; then $|h|_H \geq x_h^*(h) > 0$, as x_h^* is non-constant. This is an analogue of the Gelfand-Raĭkov theorem on point separation by characters [HewR, 22.12], cf. [Tar]. See [Bad, Lemma 1] for a variant Hahn-Banach extension theorem in the present commutative group context.) It is not clear, however, whether the resulting group is trivial for $\mu \in \mathcal{P}(X)$. In the classical Gaussian γ context, $|h|_H = \infty$ implies the mutual singularity $\gamma_h \perp \gamma$ [Bog1, 2.4.5(i)].

For $h \in H$ and $n \in \mathbb{N}$, assuming w.l.o.g. that the supremum for $|h|_H$ occurs with $x^*(h) > 0$,

$$\begin{aligned} |nh|_H &= \sup\{x^*(nh) : x^* \in X^*, R(x^*, x^*) \leq 1\} \\ &= \sup\{nx^*(h) : x^* \in X^*, R(x^*, x^*) \leq 1\} \\ &= n|h|_H. \end{aligned}$$

Thus the norm $|.|_H$ (which is subadditive) may be said to be $\mathbb{N}$-*homogeneous*, as in [BinO2, §3.2], or *sublinear* in the sense of Berz (see below).

Suppose now that the group X is (infinitely) divisible, so x/n is defined for $n \in \mathbb{N}$, as is also qx for rational q. It follows by a similar argument to that above that if $h \in H$ and $q > 0$, then $|qh|_H < \infty$, and so H is also divisible. We now briefly study sublinear group-norms on a divisible abelian group, proving in particular in Prop. 5.1 that H is a topological vector space.

Definition. In an abelian ($\mathbb{N}$-) divisible group G, since its group-norm $||.||$ is subadditive, we follow Berz [Ber] (cf. [BinO3,5,6]) in calling it *sublinear* if

$$||ng|| = n||g|| \qquad (g \in G, n \in \mathbb{N})$$

(so that $||g/n|| = ||g||/n$), or equivalently and more usefully:

$$||qg|| = q||g|| \qquad (g \in G, q \in \mathbb{Q}_+).$$

Remark. The triangle inequality for the norm gives $||g|| \leq n||g/n||$; the definition requires the reverse inequality.

Proposition 5.1. *A divisible abelian topological group complete under a sublinear group-norm is a topological vector space under the action*

$$t \cdot g := \lim_{q \to t} qg \qquad (t \in \mathbb{R}_+,\ q \in \mathbb{Q}_+,\ g, \in G). \tag{†}$$

In particular, the group has the embeddability property defined by $t \mapsto t \cdot g$, for $t \in \mathbb{R}_+$.

Proof. Since

$$||(q - q')(g - g')|| = |q - q'| \cdot ||g - g'|| \qquad (q, q' \in \mathbb{Q}_+,\ g, g' \in G),$$

and the group is norm-complete, the action (†) is well defined. It is also jointly continuous, since passing to appropriate limits over $\mathbb{Q}_+$

$$||(t - t')(g - g')|| = |t - t'| \cdot ||g - g'|| \qquad (t, t' \in \mathbb{R}_+,\ g, g' \in G).$$

The action extends to $\mathbb{R}$ by taking $-t \cdot g := t \cdot (-g)$, and converts the group into a vector space, as $(q + q')g = qg + q'g$ and $q(g + g') = qg + qg'$ may be taken to the limit through $\mathbb{Q}$. Likewise the final statement follows, as

$$||t \cdot g|| = \lim_{q \to t} ||qg|| = t||g||. \qquad \square$$

We verify that the classical boundedness theorem holds in the group context for additive functions, and hence that such functions are linear (in the sense of the action (†)).

Proposition 5.2. *For an abelian divisible group G complete under a sub-linear group-norm and $x^* : G \to \mathbb{R}$ additive, x^* is continuous iff*

$$||x^*|| = \sup\{|x^*(x)|/||x|| : x \in G\backslash\{0\}\} < \infty,$$

and then

$$|x^*(x)| \leq ||x^*|| \cdot ||x||.$$

Proof. If x^* is continuous, choose $\delta > 0$ with $|x^*(x)| \leq 1$ for $||x|| \leq \delta$. For $x \neq 0$ and rational $q \geq ||x||/\delta$, as $||x/q|| = ||x||/q \leq \delta$,

$$|x^*(x)| = q|x^*(x/q)| \leq q;$$

then taking limits as $q \to ||x||/\delta$ through $\mathbb{Q}$ yields

$$|x^*(x)| \leq ||x||/\delta.$$

This holds also for $x = 0$. So $||x^*|| \leq 1/\delta < \infty$. In this case, by definition, $|x^*(x)| \leq ||x^*|| \cdot ||x||$ for $x \neq 0$, and this again holds also for $x = 0$.

Conversely, if $||x^*|| < \infty$, then again $|x^*(x)| \leq ||x^*|| \cdot ||x||$, and so x^* is continuous at 0 and hence everywhere. □

Corollary 5.1. *A continuous additive function x^* may be extended by taking*

$$x^*(t \cdot g) := \lim_{q \to t} x^*(qg) \qquad (g \in G,\ q \in \mathbb{Q}_+,\ t \in \mathbb{R}_+),$$

which then makes it linear in the sense of the action $t \cdot g$:

$$x^*(t \cdot g) = t \cdot x^*(g) \qquad (g \in G, t \in \mathbb{R}_+).$$

Proof. Convergence of $x^*(qg)$ as $q \to t$ follows from the finiteness of $||x^*||$ and

$$|x^*(qg) - x^*(q'g)| = |x^*((q - q')g)| \leq ||x^*|| \cdot |q - q'| \cdot ||g||.$$

The final claim follows, taking limits through $\mathbb{Q}_+$,

$$x^*(tg) = \lim_{q \to t} x^*(qg) = \lim_{q \to t} q \cdot x^*(g) = t \cdot x^*(g). \qquad \square$$

Returning to the context of the Cameron-Martin group-norm, notwithstanding the issue of possible triviality of H above, one may define here as in the classical context the set $X_\mu^* \subseteq L^2(\mu)$, by passing to the closed span in $L^2(\mu)$ of the centred (mean-zero) image $\{x^* - \mu(x^*) : x^* \in X^*\}$ of X^*. Then one may extend the domain of R above to the subspace X_μ^* of $L^2(\mu)$, so that $R(f)$ for $f \in X_\mu^*$ is given by

$$R(f)(y^*) := \int f(x)[y^*(x) - \mu(y^*)]d\mu(x) \qquad (y^* \in X^*)$$

(as above, as we are centering at means here). Then $h \in H$ if $h = R(g)$ for some $g = \hat{h} \in X_\mu^*$; then $\delta_h(y^*) = y^*(h) = R(g)(y^*)$, for $y^* = z^* - \mu(z^*)$ with $z^* \in X^*$, and so taking $y^* \to g$

$$|h|_H = ||g||_{L^2(\mu)}.$$

Conversely, for fixed $h \in H$, the map $f \mapsto f(h)$, for $f \in X_\mu^*$ (including $f = x^* - \mu(x^*)$ for $x^* \in X^*$) is represented by

$$\langle f, \hat{h} \rangle_{L^2(\mu)},$$

by the Riesz Representation Theorem. So $h = R(\hat{h})$.

For $h, k \in H$, define their inner product by referring to the elements $\hat{h}, \hat{k}$ such that $h = R(\hat{h})$, $k = R(\hat{k})$, passing in R to the limit, and putting

$$(h, k)_H := \int_X \hat{h}(x)\hat{k}(x)d\mu(x).$$

Then H inherits an inner-product structure by

$$|h|_H^2 = ||\hat{h}||_{L^2(\mu)}^2 = \int_X (\hat{h}(x))^2 d\mu(x) = (h, h)_{H(\mu)}.$$

In the vector-space Gaussian context, for $h \in H$ the density of μ_h w.r.t. μ is given by (CM) above in §1.

For Gaussian measures on (locally compact) groups G, see §9.12.

6 Reference measures with selective subcontinuity

We briefly recall the construction of a measure due to Solecki, as this exemplifies a property at the heart of the construction of a Steinhaus triple in §7.

For this it will be helpful to review from Th. 3.4 the notation $\mu_{-}^{\mathbf{t}}$ for $\mathbf{t} = \{t_n\}$ a null sequence, i.e. with $t_n \to 1_G$:

$$\mu_{-}^{\mathbf{t}}(K) := \liminf_{n\to\infty} \mu(Kt_n) \qquad (K \in \mathcal{K}(G)).$$

When $\mu(K) = \mu_{-}^{\mathbf{t}}(K)$ we refer to *selective subcontinuity* of μ 'along' $\mathbf{t}$ (cf. [BinO7]). Thus, for G not locally compact and $K \in \mathcal{K}_{+}(\mu)$, by Cor. 3.2 there will be $\mathbf{t}$ with $\mu(Kt_n) \to 0$, but there may, and in certain specified circumstances necessarily will, also exist $\mathbf{t}$ with $\mu_{-}^{\mathbf{t}}(K) > 0$ (as when the Subcontinuity Theorem, Th. 6.1 below, holds).

Recall that a (Polish) group G is *amenable at* 1 ([Sol]; cf. [BinO7, §2] for the origin of this term) if, given any sequence of measures $\nu_n \in \mathcal{P}(G)$ with $1_G \in \mathrm{supp}(\nu_n)$, there are σ and σ_n in $\mathcal{P}(G)$ with $\sigma_n \ll \nu_n$ for each $n \in \mathbb{N}$, and

$$\sigma_n * \sigma(K) \to \sigma(K) \qquad (K \in \mathcal{K}(G)).$$

Here $*$ again denotes convolution. (Abelian Polish groups all have this property [Sol, Th. 3.2].) With δ_g the Dirac measure at g and $\mathbf{t}$ a non-trivial null sequence, taking

$$\nu_n := 2^{n-1} \sum\nolimits_{m\geq n} 2^{-m} \delta_{t_m^{-1}} \in \mathcal{P}(G)$$

, we denote by $\sigma_n(\mathbf{t})$ and $\sigma(\mathbf{t})$ the measures whose existence the definition above asserts. We term $\sigma(\mathbf{t})$ a *Solecki reference measure*. Below we place further restrictions on the rate of convergence of $\mathbf{t}$ and refer to a symmetrized version of ν_n.

In the above setting, $\sigma(K) > 0$ implies that for some sequence $m(n) := m_K(n)$

$$\sigma(Kt_{m_K(n)}) \to \sigma(K).$$

We repeat here a result from the companion paper [BinO7], as its proof motivates the definition which follows.

Theorem 6.1 (**Subcontinuity Theorem**, after Solecki [Sol, Th. 1(ii)]). *For G amenable at 1, $0 < \theta < 1$, and $\mathbf{t}$ a null sequence, there is $\sigma = \sigma(\mathbf{t}) \in \mathcal{P}(G)$ such that for each $g \in G, K \in \mathcal{K}(G)$ with $\sigma(gK) > 0$ there is a subsequence $\mathbf{s} = \mathbf{s}(g, K) := \{t_{m(n)}\}$ with*

$$\sigma(gKt_{m(n)}) > \theta\sigma(gK) \qquad (n \in \mathbb{N}), \text{ so} \qquad \sigma_{-}^{\mathbf{s}}(gK) > 0.$$

That is, σ is subcontinuous along $\mathbf{s}$ *on gK. In particular*

$$\sigma(K) = \sigma^{\mathbf{t}}_{-}(K).$$

Proof. For $\mathbf{t} = \{t_n\}$ null, put $\nu_n := 2^{n-1} \sum_{m \geq n} 2^{-m} \delta_{t_m^{-1}} \in \mathcal{P}(G)$; then $1_G \in \mathrm{supp}(\nu_n) \supseteq \{t_m^{-1} : m > n\}$. By definition of amenability at 1, in $\mathcal{P}(G)$ there are σ and $\sigma_n \ll \nu_n$, with $\sigma_n * \sigma(K) \to \sigma(K)$ for all $K \in \mathcal{K}(G)$.

Fix $K \in \mathcal{K}(G)$ and g with $\sigma(gK) > 0$. As gK is compact, $\sigma_n * \sigma(gK) \to \sigma(gK)$; then w.l.o.g.

$$\nu_n * \sigma(gK) > \theta\sigma(gK) \qquad (n \in \mathbb{N}). \tag{‡}$$

For $m, n \in \mathbb{N}$ choose $\alpha_{mn} \geq 0$ with $\sum_{m \geq n} \alpha_{mn} = 1$ $(n \in \mathbb{N})$ and $\nu_n := \sum_{m \geq n} \alpha_{mn} \delta_{t_m^{-1}}$. Then for each n there is $m = m(n) \geq n$ with

$$\sigma(gKt_m) > \theta\sigma(gK);$$

otherwise, summing the reverse inequalities over all $m \geq n$ contradicts (‡). So $\lim_n \sigma(gKt_{m(n)}) \geq \theta\sigma(gK) : \sigma$ is subcontinuous along $\mathbf{s} := \{t_{m(n)}\}$ on gK.

For the last assertion, take $g = 1_G$ and recall that $\sigma(K) \geq \sigma^{\mathbf{t}}_{-}(K)$, by upper semi-continuity of $t \mapsto \sigma(Kt)$. □

A corollary to this is a 'non-Haar-null' version of the Steinhaus-Weil interior point theorem:

Theorem SWbH (Steinhaus-Weil Theorem beyond Haar [Sol, Th 1(ii)]). *For G amenable at 1 : if $E \in \mathcal{U}(G)$ is not left Haar null, then* $1_G \in \mathrm{int}(E^{-1}E)$.

Proof. Suppose otherwise; then $1_G \notin \mathrm{int}(E^{-1}E)$ and we may choose $t_n \in B_{1/n} \backslash E^{-1}E$. As $t_n \to 1_G$, choose $\sigma = \sigma(\mathbf{t})$ as in the preceeding theorem. Since E is not left Haar null, $\sigma(gE) > 0$ for some g. For this g, choose compact $K \subseteq E$ with $\sigma(gK) > 0$. Then by the Subcontinuity Theorem (Th. 6.1) and by Prop. 3.1 for $\Delta = \sigma^{\mathbf{t}}_{-}(K)/4$, there is $\delta > 0$ such that

$$\emptyset \neq B^{\Delta}_{\delta} \subseteq K^{-1}g^{-1}gK = K^{-1}K;$$

moreover, as in Prop. 3.1, $t_n \in K^{-1}K \subseteq E^{-1}E$ for infinitely many n, which contradicts the choice of $\mathbf{t}$. So $1_G \in \mathrm{int}(E^{-1}E)$. □

Definition ([BinO7]). Say that a null sequence $\mathbf{t}$ is *regular* if $\mathbf{t}$ is non-trivial, $\{||t_k||\}_k$ is non-increasing, and

$$||t_k|| \leq r(k) := 1\,/[2^k(k+1)] \qquad (k \in \mathbb{N}).$$

For regular $\mathbf{t}$, put

$$\tilde{\nu}_k = \tilde{\nu}_k(\mathbf{t}) := 2^{k-2}\sum\nolimits_{m\geq k} 2^{-m}(\delta_{t_m^{-1}} + \delta_{t_m}) = \frac{1}{4}\delta_{t_k^{-1}} + \frac{1}{4}\delta_{t_k} + \frac{1}{8}\delta_{t_{k+1}^{-1}} + \ldots.$$

Then $\tilde{\nu}_k(B_{r(k)}) = 1$ and $\tilde{\nu}_k$ is *symmetric*: $\tilde{\nu}_k(K^{-1}) = \tilde{\nu}_k(K)$, as $t \in K$ iff $t^{-1} \in K^{-1}$. In the definition below, which is motivated by the proof of Theorem 6.1, we will view the measure $\tilde{\nu}_k$ as just another version of ν_k above (by merging $\mathbf{t}^{-1}$ with $\mathbf{t}$ by alternation of terms).

Definition ([BinO7]). Say that a (Polish) group G is *strongly amenable at 1* if G is amenable at 1, and for each *regular* $\mathbf{t}$ the Solecki measure $\sigma(\mathbf{t})$ corresponding to $\nu_k(\mathbf{t})$ has associated measures $\sigma_k(\mathbf{t}) \ll \nu_k(\mathbf{t})$ with the following *concentration property*. Writing, for $k \in \mathbb{N}$,

$$\sigma_k := \sum\nolimits_{m\geq k} a_{km}\delta_{t_m^{-1}},$$

for some non-negative sequences $\mathbf{a}_k := \{a_{kk,}a_{k,k+1,}a_{k,k+2},\ldots\}$ of unit ℓ_1-norm, there is an index j and $\alpha > 0$ with

$$a_{k,k+j} \geq \alpha > 0 \qquad \text{for all large } k.$$

A refinement of Solecki's proof of the Subcontinuity Theorem (Th. 6.1 above) yields the following two results, for the proof of which we refer the reader to the companion paper [BinO7].

Theorem 6.2 (**Strong amenability at 1**, [BinO7, Th. 4] after [Sol2, Prop. 3.3(i)]). *Any abelian Polish group G is strongly amenable at 1.*

Theorem 6.1$_\mathrm{S}$ (Strong Subcontinuity Theorem). *For G a (Polish) group that is strongly amenable at 1, if $\mathbf{t}$ is regular and $\sigma = \sigma(\mathbf{t})$ is a Solecki measure – then for $K \in \mathcal{K}_+(\sigma)$*

$$\sigma(K) = \lim_n \sigma(Kt_n) = \sigma_-^{\mathbf{t}}(K).$$

Remark. The reference measure $\sigma(\mathbf{t})$ in the last theorem may in fact be selected symmetric [BinO7], in which case $Q_R(\sigma(\mathbf{t}))$ is a subgroup.

We note an immediate corollary, useful in §8 below.

Corollary 6.1. *For $G, \mathbf{t}$ and σ as in Th. 6.1$_{\mathrm{S}}$ above, and $K, H \in \mathcal{K}(G), \delta > 0$: if $0 < \Delta < \sigma(K)$ and $0 < D < \sigma(H)$, then there is n with*

$$B_\delta^{K\Delta} \cap B_\delta^{HD} \supseteq \{t_m : m \geq n\}.$$

Proof. Take $\varepsilon := \min\{\sigma(K) - \Delta, \sigma(H) - D\} > 0$. As $K, H \in \mathcal{K}_+(\sigma)$, there is n such that $||t_m|| < \delta$ for $m \geq n$ and

$$\sigma(Kt_m) \geq \sigma(K) - \varepsilon \geq \Delta, \qquad \sigma(Ht_m) \geq \sigma(H) - \varepsilon \geq D \qquad (m \geq n). \qquad \square$$

7 The Steinhaus support of a measure

In this section we construct (one might say via a 'disaggregation') the Steinhaus support $H(\mu)$ of a probability measure μ defined on a Polish group G (see Th. 7.1); this is possible provided the measure has 'sufficient subcontinuity' (defined below) – sufficient to allow a relative-interior Steinhaus property, relative to some embedded 'subspace'. In the next section we apply the construction to a Solecki measure $\sigma(\mathbf{t})$ for a regular null sequence $\mathbf{t}$ as in §6.

Definition. Say that a probability measure has *sufficient subcontinuity*, written $\mu \in \mathcal{P}_{\mathrm{suf}}(G)$, if for all $K \in \mathcal{K}_+(\mu)$ and $\delta > 0$ there is $\Delta(K, \delta) \geq 0$ so small that for $\Delta(K, \delta) < \Delta < \mu(K)$

$$B_\delta^{K,\Delta} = B_\delta^{K,\Delta}(\mu) = \{s \in B_\delta : \mu(Ks) > \Delta\}$$

is infinite. Above, if $\Delta(K, \delta) \equiv 0$, say that μ is of *Solecki type*; these will be considered in §8.

Lemma 7.1 below asserts that, for G amenable at 1, a Solecki measure $\mu = \sigma(\mathbf{t})$ has this property. Further motivation for working with this assumption is provided by Th 3.6: for (H, G, μ) a Steinhaus triple and $K \in \mathcal{K}_+(\mu)$, the set $\mathcal{O}(K) := \{s \in H : \mu(Ks) > 0\}$ is open in H and $\{s \in H : \mu(Ks) = \mu_-^H(Ks) > 0\}$ is dense in $\mathcal{O}(K)$.

The goal here is to create a new topology, if not on G then on a dense subset of G, in which the sets

$$B_\delta^{K,\Delta} := \{z \in B_\delta : \mu(Kz) > \Delta\}$$

(with μ understood from context) shall be open sub-base members for selected $K \in \mathcal{K}_+(\mu)$. This is tantamount to requiring that the corresponding maps $z \mapsto \mu(Kz)$ be continuous on some subset of G; cf. Th. 3.2′ , also by way of justification.

Whenever we consider sets $B_\delta^{K,\Delta}$ for $K \in \mathcal{K}_+(\mu)$ and $\delta > 0$ we implicitly assume that $\Delta \geq \Delta(K, \delta)$.

Notice the monotonicities:

$$\Delta \leq D \Longrightarrow B_\delta^{K,D} \subseteq B_\delta^{K,\Delta}, \qquad \eta \leq \varepsilon \Longrightarrow B_\eta^{K,\Delta} \subseteq B_\varepsilon^{K,\Delta}.$$

The sequence of Lemmas 7.1-7.4 below justifies the introduction of a new topology with sub-basic sets of the form $gB_\delta^{K,\Delta}$, but *only* on those points of G that can be *covered* by these sets: the detailed statement is in Theorem 7.1 below. The proof strategy demands both a countable iteration – an inductive generation of a family of sets $B_\delta^{K,\Delta}$ – and then a countable subgroup of translators g. In the subsequent section, we identify which are the points that can be covered.

For $\mu \in \mathcal{P}(G), \mathcal{H} \subseteq \mathcal{K}_+(\mu)$, and $\mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$, we put

$$\mathcal{B}_{1_G}(\mu, \mathcal{H}) := \{B_\delta^{K,\Delta}(\mu) : K \in \mathcal{H};\ \delta, \Delta \in \mathbb{Q}_+;\ 0 < \Delta < \mu(K)\};$$

this is to be a neighbourhood base at 1_G. For $\mathcal{H} = \mathcal{K}_+(\mu)$ we abreviate this to $\mathcal{B}_{1_G}(\mu)$ or even to $\mathcal{B}_{1_G}$, when μ is understood.

Lemma 7.1. *For $\mu \in \mathcal{P}(G)$, $\mathbf{t}$ null and non-trivial, and arbitrary $\delta > 0$: if $0 < \Delta < \mu_-^{\mathbf{t}}(K)$, then $B_\delta^{K,\Delta}(\mu) \neq \{1_G\}$. In particular, for G amenable at 1 and $\mu = \sigma(\mathbf{t})$: if $0 < \Delta < \sigma(K)$, then $B_\delta^{K,\Delta}(\sigma)$ is infinite.*

Proof. Since $\mathbf{t}$ is null and non-trivial, for all large enough n both $t_n \in B_\delta$ and also $\mu(Kt_n) > \Delta$. For $\mu = \sigma(\mathbf{t})$ and $0 < \Delta < \sigma(K)$, pick $0 < \theta < 1$ with

$$\theta\sigma(K) = \Delta.$$

Then for some, necessarily non-trivial, subsequence $\mathbf{s} := \{s_n\}$ of $\mathbf{t}$,

$$\sigma(Ks_n) > \theta\sigma(K) = \Delta.$$

So $B_\varepsilon^{K,\Delta} = \{s \in B_\varepsilon : \sigma(Ks) > \Delta\}$ is infinite. □

Lemma 7.2. *For $\mu \in \mathcal{P}_{\rm suf}(G)$ and $K \in \mathcal{K}_+(\mu)$: if $w \in B_\delta^{K,\Delta}$, then for $H = Kw$ and some $\varepsilon > 0$*

$$\{w\} \neq wB_\varepsilon^{H,\Delta} \subseteq B_\delta^{K,\Delta}.$$

In particular:
(i) *if $1_G \in gB$ for some $B \in \mathcal{B}_{1_G}$, then there is is $B' \in \mathcal{B}_{1_G}$ with $1_G \in B' \subseteq gB$;*
(ii) *for G amenable at 1: if $\mu = \sigma(\mathbf{t})$ with $\mathbf{t}$ null, then $B_\varepsilon^{H,\Delta}$ may be selected infinite.*

Proof. As $w \in B_\delta$ there is $0 < \varepsilon < \delta$ with $wB_\varepsilon \subset B_\delta$. As $w \in B_\delta^{K,\Delta}$, $\mu(H) = \mu(Kw) > \Delta$, so, passing to a smaller ε if necessary, there is $\Delta(H, \varepsilon) > 0$ so that $B_\varepsilon^{H,\Delta'}$ is infinite for $\Delta' \geq \max\{\Delta(H, \varepsilon), \Delta\} > \Delta(K, \delta)$. Then

$$\begin{aligned} w &\in wB_\varepsilon^{H,\Delta'} = w\{s \in B_\varepsilon : \mu(Hs) > \Delta'\} = \{ws \in wB_\varepsilon : \mu(Kws) > \Delta'\} \\ &\subseteq \{x \in B_\delta : \mu(Kx) > \Delta\} = B_\delta^{K,\Delta}. \end{aligned}$$

For the last part, suppose $1_G \in gB$ with $B = B_\delta^{K,\Delta} \in \mathcal{B}_{1_G}$; then $w \in B$ for $w = g^{-1}$. Applying the first part, take $B' := B_\varepsilon^{H,\Delta} \in \mathcal{B}_1$ for $H = Kw$ and the $\varepsilon > 0$ above; then,

$$w \in wB' = B_\varepsilon^{H,\Delta} \subseteq B_\delta^{K,\Delta} = B : \qquad 1_G \in B' \subseteq gB. \qquad \square$$

Corollary 7.1. *If $x \in yB \cap zC$ for $x, y, z \in G$ and some $B, C \in \mathcal{B}_{1_G}$, then $x \in x(B' \cap C') \subseteq yB \cap zC$ for some $B', C' \in \mathcal{B}_{1_G}$.*

Proof. As $1_G \in x^{-1}yB$ and $1_G \in x^{-1}zC$, there are $B', C' \in \mathcal{B}_{1_G}$ with $1_G \in B' \subseteq x^{-1}yB$ and $1_G \in C' \subseteq x^{-1}zC$. Then $x \in xB' \cap xC' \subseteq yB \cap zC$. □

We now improve on Lemma 7.2 by including some technicalities, whose purpose is to introduce a *separable* topology on a subspace of G refining that induced by τ_G. In view of the monotonicities observed above, we may restrict attention to $\delta, \Delta \in \mathbb{Q}_+$.

Lemma 7.3. *For $\mu \in \mathcal{P}_{\mathrm{suf}}(G)$ and countable $\mathcal{H} \subseteq \mathcal{K}_+(\mu)$, there is a countable $D = D(\mathcal{H}) \subseteq G$ accumulating at 1_G such that: if $w \in B_\delta^{K,\Delta}$ with $K \in \mathcal{H}$, $\delta, \Delta \in \mathbb{Q}_+$ and $\Delta < \mu(K)$, then for some $g \in D$ with $\mu(Kg) > \Delta$ and some $\varepsilon \in \mathbb{Q}_+$,*

$$w \in gB_\varepsilon^{Kg,\Delta} \subseteq B_\delta^{K,\Delta}.$$

Proof. As G is separable, we may choose $\{\bar{g}_m\}_{m\in\mathbb{N}} = \{\bar{g}_m(B_\delta^{K,\Delta})\}_{m\in\mathbb{N}} \subseteq B_\delta^{K,\Delta}$ dense in $B_\delta^{K,\Delta}$, an infinite set, by Lemma 7.1. Take

$$D = D(\mathcal{H}) := \{\bar{g}_m(B_\delta^{K,\Delta}) : K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+, \Delta < \mu(K)\},$$

which is countable. Since $B_\delta^{K,\Delta} \subseteq B_\delta$, D accumulates at 1_G. We claim that D above satisfies the conclusions of the Lemma.

Fix $w \in B_\delta^{K,\Delta}$, with K, Δ, δ as in the hypotheses. Choose $\varepsilon \in \mathbb{Q}_+$ with

$$wB_{3\varepsilon} \subseteq B_\delta.$$

Choose $\bar{g}_m = \bar{g}_m(B_\delta^{K,\Delta})$ with $||\bar{g}_m^{-1}w|| < \varepsilon$, possible by construction of $\{\bar{g}_m(B_\delta^{K,\Delta})\}$. Put $z_m := \bar{g}_m^{-1}w$; then $w = \bar{g}_m z_m$, $z_m \in B_\varepsilon$ and $\bar{g}_m \in wB_\varepsilon$, so $w \in \bar{g}_m z_m B_\varepsilon \in \bar{g}_m B_{2\varepsilon} \subseteq wB_{3\varepsilon} \subseteq B_\delta$. By choice of $\{\bar{g}_m\}_{m\in\mathbb{N}}$, $\mu(K\bar{g}_m) > \Delta$, and furthermore

$$\begin{aligned} w &\in \bar{g}_m z_m\{s \in B_\varepsilon : \mu(K\bar{g}_m z_m s) > \Delta\} \subseteq \bar{g}_m\{t \in B_{2\varepsilon} : \mu(K\bar{g}_m t) > \Delta\} \\ &= \bar{g}_m B_{2\varepsilon}^{K\bar{g}_m,\Delta} \subseteq \{x \in B_\delta : \mu(Kx) > \Delta\} = B_\delta^{K,\Delta}, \end{aligned}$$

as $\bar{g}_m B_{2\varepsilon} \subseteq B_\delta$. $\square$

In Lemma 7.3 above Kg need not belong to $\mathcal{H}$. Lemma 7.4 below asserts that Lemma 7.3 holds on a countable family $\mathcal{H}$ of compact sets that is closed under the appropriate translations.

Lemma 7.4. *For $\mu \in \mathcal{P}_{\mathrm{suf}}(G)$, there are a countable $\mathcal{H} \subseteq \mathcal{K}_+(\mu)$ and a countable set $D = D(\mathcal{H}) \subseteq G$ dense in G such that: if $w \in B_\delta^{K,\Delta}$ with $K \in \mathcal{H}$, $\delta, \Delta \in \mathbb{Q}_+$ and $0 < \Delta < \mu(K)$, then for some $g \in D$ with $\mu(Kg) > \Delta$, $Kg \in \mathcal{H}$ and some $\varepsilon \in \mathbb{Q}_+$,*

$$w \in gB_\varepsilon^{Kg,\Delta} \subseteq B_\delta^{K,\Delta}.$$

Proof. Suppose μ is concentrated on $\bigcup_n K_n$, with the K_n compact and $\mu(K_n) > 0$. Taking $\mathcal{H}_0$ to comprise the basic compacts $K_n \cap g_m\bar{B}_\delta$ with $\{g_m\}$ dense in G and $\delta \in \mathbb{Q}_+$, proceed by induction:

$$\mathcal{H}_{n+1} := \{Kg : K \in \mathcal{H}_n, g \in D(\mathcal{H}_n), \delta, \Delta \in \mathbb{Q}_+,\ 0 < \Delta < \mu(K)\},$$

$$\mathcal{H} := \bigcup\nolimits_n \mathcal{H}_n, \qquad D := \bigcup\nolimits_n D(\mathcal{H}_n). \qquad \square$$

Theorem 7.1. *For $\mu \in \mathcal{P}_{\rm suf}(G)$ there are a countable $\mathcal{H} \subseteq \mathcal{K}_+(\mu)$ and a countable set $\Gamma = \Gamma(\mathcal{H}) \subseteq G$ dense in G such that, taking*

$$\mathcal{B}_{\mathcal{H}}(\mu) := \{B_\delta^{K,\Delta}(\mu) \in \mathcal{B}_{1_G} : K \in \mathcal{H}; \delta, \Delta \in \mathbb{Q}_+;\ 0 < \mu(K) < \Delta\},$$

$$\mathcal{B}(\mu) = \mathcal{B}_\Gamma(\mu) := \Gamma \cdot \mathcal{B}_{\mathcal{H}}(\mu) = \{\gamma B : \gamma \in \Gamma, B \in \mathcal{B}_{\mathcal{H}}(\mu)\}$$

is a sub-base for a second-countable topology on the subset

$$H(\mu) := \bigcup \mathcal{B}_\Gamma(\mu) = \bigcup \{\gamma B : B \in \mathcal{B}_{\mathcal{H}}(\mu), \gamma \in \Gamma\}.$$

Proof. Take a countable subgroup Γ in G, which is dense in G under τ_G and contains $D(\mathcal{H})$, as in Lemma 7.4. Consider $w \in \gamma B_\delta^{K,\Delta}$ with $\gamma \in \Gamma$, $K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+,\ \Delta < \mu(K)$. Then for some $g \in D(\mathcal{H}) \cap B_\delta^{K,\Delta}$ and $\varepsilon > 0$

$$gB_\varepsilon^{Kg,\Delta} \subseteq B_\delta^{K,\Delta}.$$

So both

$$w \in \gamma g B_\varepsilon^{Kg,\Delta} \subseteq \gamma B_\delta^{K,\Delta},$$

and $\gamma g \in \Gamma$, the latter as $g \in D(\mathcal{H}) \subseteq \Gamma$. So, by the Corollary 7.1 (of Lemma 7.2), the family $\mathcal{B}(\mu)$ forms a sub-base for a topology on the set of points

$$\bigcup \{\gamma B : B \in \mathcal{B}(\mu), g \in \Gamma\}. \qquad \square$$

Remark. The same proof shows that one may drop countability in the conditions and second-countability in the conclusions.

Definition. We term the second-countable topology of the preceeding theorem the *μ-topology.*

In the next result we take $\mu = \sigma(\mathbf{t})$. As $1_G \in B_\delta^{K,\Delta} \subseteq B_\delta$, the $\sigma(\mathbf{t})$-topology evidently refines the *original topology* τ_G of G. The finer topology could be discrete; in cases of interest, however, this will not happen:

Proposition 7.1 (Refinement). *For G amenable at 1, $\mathbf{t}$ null and non-trivial, the open sets $B_\delta^{K,\Delta}$ of the $\sigma(\mathbf{t})$-topology are infinite and refine the topology induced by τ_G on $H(\sigma(\mathbf{t}))$.*

Proof. For $\{g_n\}_{n\in\mathbb{N}}$ dense in G, write $D := \{g_n : n \in \mathbb{N}\}$. The open sets of G are generated as unions of sets of the form gV, with $g \in D$ and V an open nhd of 1_G. We show that these sub-basic sets of the $\sigma(\mathbf{t})$-topology refine the τ_G-nhds of the identity. For V a non-empty τ_G-open nhd of 1_G choose U to be a non-empty τ_G-open nhd of 1_G, with $U^{-1}U \subseteq V$.

Consider any non-trivial null sequence $\mathbf{t}$ and, referring to the Subcontinuity theorem (Th. 7.1), consider $\sigma = \sigma(\mathbf{t}) \in \mathcal{P}(G)$. For $\{g_n\}$ dense in G, there is n with $\sigma(g_nU) > 0$; for otherwise, $\sigma(g_nU) = 0$ for each n and, since $G = \bigcup_n g_nU$, we reach the contradiction $\sigma(G) = 0$. Pick n with $\sigma(g_nU) > 0$; write g for g_n.

Choose compact sets K_n such that σ is concentrated on $\bigcup_n K_n$ and a countable base $\mathcal{B}$ for τ_G. Since

$$\sigma(gU) = \sigma(\bigcup_m K_m \cap gU) = \sigma(\bigcup_m \{K_m \cap g\bar{B} : \bar{B} \subseteq U, B \in \mathcal{B}, m \in \mathbb{N}\}),$$

there are $m \in \mathbb{N}$ and $B \in \mathcal{B}$ with $\mu(K_m \cap g\bar{B}) > 0$.

Take $K := K_m \cap g\bar{B}$; there is a subsequence $\mathbf{s} = \mathbf{s}(K) := \{t_{m(k)}\}$ with

$$\sigma(Kt_{m(k)}) > \sigma(K)/2 \qquad (k \in \mathbb{N}), \text{ so } \qquad \mu_-^{\mathbf{s}}(K) > 0.$$

So as $\Delta := \sigma_-^{\mathbf{s}}(K)/4 < \sigma_-^{\mathbf{s}}(K) < \sigma(K)$, by [BinO7, Lemma 1], there is $\delta > 0$ with

$$1_G \in B_\delta^{K,\Delta} \subseteq K^{-1}K \subseteq \bar{B}^{-1}g^{-1}g\bar{B} \subseteq U^{-1}U \subseteq V,$$

and $t_{m(n)} \in B_\delta^{K,\Delta}$ for all large enough n as in [BinO7, Lemma 1]. $\square$

Remark. Proposition 7.1 is connected to the Steinhaus-Weil Theorem, Theorem SW, in §6 above: a similar argument gives, for E non-left-Haar-null, that $1_G \in \mathrm{int}_G(\hat{E})$ for

$$\hat{E} := \bigcup_{\delta,\Delta>0,g\in G,\mathbf{t}} \{B_\delta^{gK,\Delta}(\sigma(\mathbf{t})) : K \subseteq E, K \in \mathcal{K}_+(\sigma(\mathbf{t})), \Delta < \sigma(gK)\}.$$

That is, the relevant basic open nhds of 1_G in the various $\sigma(\mathbf{t})$-topologies 'aggregate' to yield a nhd of 1_G in the original topology of G.

8 Connections with Cameron-Martin theory

In this section, we pursue the connection with Cameron-Martin theory. Proposition 8.1 provides the basis for a definition of the 'covered points' under μ;

this identifies a canonical 'largest' Steinhaus support for μ (modulo an initial choice of dense subset). The result takes its motivation from the following classical observation:

In a locally convex topological vector space X, in particular in a Fréchet space, equipped with a symmetric Radon Gaussian measure γ: if E is any Hilbert space continuously embedded in $H(\gamma)$, then there exists a symmetric Radon Gaussian measure γ' with $H(\gamma') = E$ [Bog1, 3.3.5].

We sign off by showing that the topology of the Steinhaus support is metrizable.

We recall from §4.2: $\mu_g(B) = \mu(Bg)$ for $B \in \mathcal{B}(G)$;

$$\mathcal{N}=\mathcal{N}(\mu) := \{g \in G : \mu_g \perp \mu\}, \qquad G_0 = G_0(D) := \bigcup\nolimits_{d\in D} d\mathcal{N}(\mu),$$

with $D = \{g_n : n \in \mathbb{N}\}$ a dense subset of G; measures of Solecki type (§7) have $\Delta(K, \delta) \equiv 0$.

Proposition 8.1 (Covering Lemma). *For $\mu \in \mathcal{P}(G)$ of Solecki type, let $\tilde{D}$ be a dense subset of $Q(\mu)$. For $\delta > 0, K \in \mathcal{K}_+(\mu)$ and $x \in Q(\mu)$ there is $g \in \tilde{D}$ with*

$$x \in gB_\delta^{K,\Delta}$$

for all small enough $\Delta < \mu(K)$.

Proof. Choose $g \in \tilde{D} \cap xB_\delta \subseteq Q(\mu)$. Then $y^{-1} := x^{-1}g \in B_\delta$, so also $y = g^{-1}x \in B_\delta$ (symmetry of the group-norm on G), and $y = g^{-1}x \in Q(\mu)$, as $Q(\mu)$ is a subgroup. Now $\mu(Ky) > 0$, as $y \in Q(\mu)$, so we may choose $0 < \Delta < \min\{\mu(Ky), \mu(K))$; then

$$x = gy \in g\{z \in B_\delta : \mu(Kz) > \Delta\} = gB_\delta^{K,\Delta}. \qquad \square$$

Proposition 8.1 above identifies how points of $Q(\mu)$ can be covered by certain translates of basic sets of the form $B_\delta^{K,\Delta}$. To go beyond $Q(\mu)$ this motivates the following.

Definition. Say that $g \in G$ is a *covered* point (g 'is covered') under $\mu \in \mathcal{P}(G)$ if there is $K \in \mathcal{K}_+(\mu)$ with $\mu(Kg) > 0$. (Then $g \in B_\delta^{K,\Delta}$ for $\delta > ||g||$ and $0 < \Delta < \min\{\mu(K), \mu(Kg)\}$.) So the points of Q_R are covered, but $g \in G$ is *not covered* if $\mu_g(K) = \mu(Kg) = 0$ for all $K \in \mathcal{K}_+(\mu)$, that is, $\mu_g \perp \mu$.

Proposition 8.2. *For $\mu \in \mathcal{P}(G)$ of Solecki type, $\{g_n\}_{n\in\mathbb{N}}$ dense in G and $\delta > 0$, the sets $g_n B_\delta^{K,\Delta}$ cover $G\backslash G_0 = G\backslash \bigcup_{n\in\mathbb{N}} g_n\mathcal{N}(\mu)$ and so generate a topology on the Borel set $G\backslash G_0$ for which these are sub-basic.*

Proof. Consider $x \notin \bigcup_{n\in\mathbb{N}} g_n\mathcal{N}(\mu)$ with $\{g_n\}$ is dense in G. Then for arbitrary $\delta > 0$, select $g_n \in B_\delta(x)$. Then $x \in B_\delta(g_n)$, and $y := g_n^{-1}x \in B_\delta\backslash\mathcal{N}(\mu)$, as $g_n^{-1}x \notin \mathcal{N}(\mu)$. Now we may choose $K \in \mathcal{K}_+(\mu)$ with $\mu(Ky) > 0$. Then for $0 < \Delta < \min\{\mu(K), \mu(Ky)\}$,

$$x = g_n y \in g_n B_\delta^{K,\Delta}.$$

That is, for $\delta > 0$, the family $\{g_n B_\delta^{K,\Delta} : K \in \mathcal{K}_+(\mu),\ 0 < \Delta < \mu(K), n \in \mathbb{N}\}$ covers $G\backslash G_0$, and so a second-countable topology is generated with sub-base the sets

$$\{g_n B_\delta^{K,\Delta} : K \in \mathcal{K}_+(\mu), 0 < \Delta < \mu(K), n \in \mathbb{N}, \delta \in \mathbb{Q}_+\}. \qquad \square$$

Remark. In the special case when $Q(\mu)$ is dense in G (for instance taking G to be $\bar{Q}$), so that also $\tilde{D}$ (in Prop. 8.1) is dense in G, Prop. 8.2 above (with $g_n = \tilde{g}_n$) follows from Proposition 8.1. Note that in this case also $G_0 = \mathcal{N}(\mu)$, since $\tilde{g}_n\mathcal{N}(\mu) \subseteq \mathcal{N}(\mu)$, by Lemma 4.2.

Definitions. For $\mu \in \mathcal{P}(G)$ and $K \in \mathcal{K}_+(\mu)$, put

$$\Delta_K(x, y) := |\mu(Kx) - \mu(Ky)| \leq 1 \qquad (x, y \in G),$$

which is a pseudometric, so that

$$\rho_K(x, y) := \max\{d_L^G(x, y), \Delta_K(x, y)\} \qquad (x, y \in G)$$

is a metric.

Proposition 8.3. *For $\mu \in \mathcal{P}(G)$ of Solecki type, $g \in G\backslash\mathcal{N}(\mu)$, $K \in \mathcal{K}_+(\mu)$ and $\varepsilon > 0$: if $0 < \varepsilon < \mu(Kg)$, then there is $\delta = \delta(\varepsilon)$ with $0 < \delta < \varepsilon$ such that*

$$gB_{\delta(\varepsilon)}^{Kg,\Delta} \subseteq B_\varepsilon^{\rho_K}(g) := \{x : \rho_K(x, g) < \varepsilon\} \subseteq gB_\varepsilon^{Kg,\Delta}, \text{ for } \Delta = \mu(Kg) - \varepsilon.$$

Hence, for any enumeration $\{K_n\}_n$ of the basic compact sets in $\mathcal{K}_+(\mu)$ comprising the family $\mathcal{H}_0$ of Lemma 7.4, the metric

$$\rho(x,y) := \sup\{d_L^G(x,y), 2^{-n}\Delta_{K_n}(x,y)\}$$

generates the μ-topology on $G\backslash G_0$.

Proof. Note that for $0 < \varepsilon < \mu(Kg)$

$$\Delta_K(x,g) < \varepsilon \text{ iff } [d_L^G(x,g) < \varepsilon \text{ and } \mu(Kg) - \varepsilon < \mu(Kx) < \mu(Kg) + \varepsilon].$$

Write $x = gh$; then $d_L^G(x,g) < \varepsilon$ is equivalent to the constraint $||h|| < \varepsilon$. As $x \to \mu(Kx)$ is upper semicontinuous, there is $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that $\mu(Kgh) < \mu(Kg) + \varepsilon$, for $h \in B_\delta$; this yields the further required constraint $\mu(Kgh) > \Delta := \mu(Kg) - \varepsilon$. The remaining assertions are now immediate. □

Remarks. 1. In the above argument $\mu(Kgh \triangle Kg) \leq 2\varepsilon$ provided $\mu(KgB_\delta) < \varepsilon$. This implies that convergence in ρ implies convergence in the Weil-like group-norm $||\cdot||_\mu^E$ of [BinO7] with $E = Kg$; indeed in the locally compact case these norms generate the Weil topology of [Wei] (cf. [Hal, §62], [HewR, §16], [Yam2, Ch. 2] and the recent [BinO7]). So the ρ-topology refines the Weil-like topology.
2. As with Theorem 7.1 above, there is an analogue, in which metrizability is dropped in favour of a uniform structure.

9 Complements

1. *Historical remarks.* The fundamental reference here is of course the first, Haar's 1933 paper in which he introduces Haar measure [Haa]. Von Neumann, in the paper (of the same journal) immediately after Haar's [Neu1], applies Haar measure for compact Lie groups to solve Hilbert's fifth problem. He follows this with two further contributions [Neu3,4]. Kakutani made extensive relevant contributions to both topological groups and to measure theory. His papers on the first appear in Volume 1 of his Selected Papers [Kak3], together with commentaries (p. 391-408) by A. H. Stone, J. R. Choksi, W. W. Comfort, K. A. Ross and J. Taylor. Here he deals with metrisation, with uniqueness of Haar measure, and (with Kodaira) on its completion regularity. His papers on the second appear in Volume 2, with commentaries (p.379-383) by Choksi, M. M. Rao, Oxtoby [Oxt3], and Ross. Here he deals (alone, with Kodaira, and with Oxtoby) on extension of Lebesgue measure, and with equivalence of infinite product measures (§9.18 below).

The other key historical references here are the Cameron-Martin papers [CamM1,2,3]; see [Bog1], [LedT], [Str] for textbook accounts.

2. *Radon measures.* We recall that on a complete separable metric space every Borel measure is Radon, i.e. has inner compact regularity ([Bog2, Vol. II, Th. 7.1.7]).

A metrizable Čech-complete space (i.e. one that is a $\mathcal{G}_\delta$ in some (any) compactification) is topologically complete, i.e. the topology may be generated by a complete metric [Eng, Th. 4.3.26]. So, in particular, if a locally compact group is metrizable, then it has a complete metric, and so every Borel measure on the group, in particular every Haar measure, is Radon. If in addition the group is separable (so Polish), then, being second-countable, it is σ-compact, and then every Haar measure is σ-finite, and so also outer regular ([Kal, Lemma 1.34], cf. [Par, Th. II.1.2] albeit for a probability measure).

In general, one may pass from a Haar measure η_X on a locally compact group X which is outer regular (i.e. Borel sets are outer η_X-approximable by open sets) to the Borel measure μ defined by

$$\mu(B) := \sup\{\eta_X(K) : K \in \mathcal{K}(X), K \subseteq B\} \quad (B \in \mathcal{B}(X)),$$

which agrees with η_X on $\mathcal{K}(X)$ and so is inner compact regular [Bog2, Th. 7.11.1]; however, μ need not be outer-regular. In applications inner compact regularity carries more advantages, hence our adoption of this property of measures.

We note some alternative usages here.

(a) For Schwartz [Sch, 1.2], a Radon measure is a locally finite, Borel measure which is inner compact regular (definition R_3); an equivalent definition includes local finiteness and couples outer regularity with inner compact regularity restricted to open sets (definition R_2). A third equivalent (definition R_1) involves both a locally finite measure M which is outer regular and m its associated essential measure (outer measure restricted to $\mathcal{B}(X)$) which is inner compact regular, the two agreeing on open sets and on sets of finite M-measure.

(b) Fremlin [Fre2, p. 15] defines Radon measures to be locally finite and inner compact regular (plus complete and locally determined [Fre1, p.13]).

(c) Heyer [Hey1, §1.1], for a locally compact group X, defines a Radon measure as a linear functional with domain the continuous complex functions with compact support in X and with a boundedness condition where the bounds correspond to the possible compact supports.

3. *Invariance beyond local compactness.* We recall our opening paragraph, which set out the contrast between the local compactness of the group setting, where one has Haar measure, and the absence of both in the Hilbert-space setting in which Cameron-Martin theory originated. We note that the invariance property of Haar measure may be extended *beyond* the locally-compact case. Nothing new is obtained in our setting of probability measures, but if one drops local finiteness, Haar-like measures of 'pathological' character can occur (§9.8 below). We quote Diestel and Spalsbury [DieS, Ch. 10], who give a textbook account of the early work of Oxtoby in this area [Oxt1]. We note in passing that this interesting paper is not cited by Oxtoby himself in either edition of his classic book [Oxt2]. We also note, as in 2(a), the use of local finiteness in Schwartz's definition of a Radon measure [Sch].

The classic case of Haar (invariant) measure is Lebesgue measure in Euclidean space. A number of authors have produced 'Lebesgue-like' extensions of Lebesgue measure from $\mathbb{R}^n$ to $\mathbb{R}^{\mathbb{N}}$; see e.g. Baker [Bak1,2], Gill and Zachary [GilZ], [Pan], Yamasaki [Yam1,2].

Admissible translators present themselves here and also in a variety of related circumstances; for a statistical setting see e.g. [Shep], and for later developments [Smol] and [Sad1].

4. *Quasi-invariance beyond local compactness.* Such questions are addressed in a vector-space context in Bogachev's book [Bog1]; see also Yamasaki [Yam1,2], Arutyunyan and Kosov [AruK] (cf. §4). For the group context, see Ludkovsky [Lud], Sadasue [Sad1,2] and the references cited there (again, cf. §4).

5. *Group representations beyond local compactness.* See Ludkovsky [Lud] for group representations, and the monograph of Banaszczyk [Bana] for Pontryagin duality in the abelian case; for a textbook treatment see [FelD]. For harmonic analysis, see Gel′fand and Vilenkin [GelV], Xia [Xia].

6. *Integration beyond local compactness.* Measure and integration are of course closely linked, in this context as in any other. For monograph accounts, see e.g. Skorohod [Sko], [Yam2], [Xia], [GilZ].

7. *Differentiation beyond local compactness.* Differentiation in infinitely many dimensions owes much to pioneering work by Fomin, and has led on to the theory of smooth measures and the Malliavin calculus. See e.g. Bogachev [Bog3], Dalecky and Fomin [DalF].

8. *The Oxtoby-Ulam Theorem* ([Oxt1, Th. 2], [DieS, Th.10.1]). This asserts that in a non-locally-compact Polish group carrying a (non-trivial, left) invariant Borel measure every nhd of the identity contains *uncountably* many *disjoint* (left) *translates* of a compact set of positive measure. Since local finiteness rules out such pathology, 'total' invariance of a Radon measure implies local compactness, hence the introduction of 'selective invariance' and 'selective approximation' (by compact sets) in the variant Steinhaus triples of §2.

9. *Invariant means.* One can deal with invariant means in place of invariant measures. This involves the theory of *amenable groups,* and amenability more generally; see Paterson [Pat], which has an extensive bibliography. There are links with Solecki's concept of *amenability at* 1 ([Sol] and §6; [BinO7]).

10. *Fréchet spaces: Gaussianity* [Bog1]. For X a locally convex topological vector space, γ a probability measure on the σ-algebra of the cylinder sets generated by X^* (the Borel sets, for X separable Fréchet, e.g. separable Banach), with $X^* \subseteq L^2(\gamma)$: then γ is called *Gaussian* on X iff $\gamma \circ \ell^{-1}$ defined by

$$\gamma \circ \ell^{-1}(B) = \gamma(\ell^{-1}(B)) \qquad (\text{Borel } B \subseteq \mathbb{R})$$

is Gaussian (normal) on $\mathbb{R}$ for every $\ell \in X^* \subseteq L^2(\gamma)$. For a monograph treatment of Gaussianity in a Hilbert-space setting, see Janson [Jan].

11. *Cameron-Martin aspects.* For X Fréchet and γ Gaussian case, when the closed span of $\{x^* - \gamma(x^*) : x^* \in X^*\}$ is infinite-dimensional, $H(\gamma)$ is γ-null in X [Bog1, Th. 2.4.7].

Furthermore, for $h \in H(\gamma)$, the Radon-Nikodym density $d\gamma_h/d\gamma$ (which is explicitly given by the Cameron-Martin formula (CM)) as a function of h is continuous on $H(\gamma)$ [Bog1, Cor. 2.4.3]. This implies, for $\mathbf{t}$ null in the $H(\gamma)$-norm (with $t_n \in H(\gamma)$) and compact K with $\gamma(K) > 0$, that $\gamma_-^{\mathbf{t}}(K) > 0$. Here, for X sequentially complete, the corresponding balls (under the $H(\gamma)$-norm) are weakly compact in X – cf. [Bog1, Prop. 2.4.6], also the Remark before Th. 3.4 above.

We note here for convenience the following properties of the Cameron-Martin space.

(i) For γ non-degenerate, $H(\gamma)$ is everywhere dense [Bog1, §3.6].

(ii) $\text{cl}_X H(\gamma)$ is of full measure [Bog1, 3.6.1].

(iii) If X_γ^* is infinite-dimensional (i.e. is not locally compact), then $\gamma(H(\gamma)) = 0$ [Bog1, 2.4.7].

(iv) For γ a Radon Gaussian measure, both of the spaces $H(\gamma)$ and $L^2(X, \gamma)$ are separable [Bog1, 3.2.7 and Cor. 3.2.8].

(v) The 'relative mobility property' (cf. §9.15 below) that $\gamma(Kh) \to \gamma(K)$ as $h \to 0$ always holds [Bog1, Th. 2.4.8] applied to 1_K (indeed, for $h \in H(\gamma)$, $d\gamma_h/d\gamma$ exists and is continuous in h, by [Bog1, 2.4.3]).

(vi) For γ Radon and X a locally convex topological vector space, the closed unit ball of $H(\gamma)$ is compact in X [Bog1, 3.2.4].

(vii) In a locally convex space, there is a sequence of metrizable compacta K_n with $\gamma(\bigcup_n K_n) = 1$ [Bog1, Th. 3.4.1].

(viii) For X a locally convex space, equipped with a symmetric Radon Gaussian measure γ: if E is any Hilbert space continuously embedded in $H(\gamma)$, then there exists a symmetric Radon Gaussian measure γ' with $H(\gamma') = E$ [Bog1, 3.3.5].

12. *Locally compact groups: Gaussianity.* For Gaussian measures on locally compact groups G, see e.g. [Par, IV.6] for G abelian and [Hey1, 5.2, 5.3] for the general case. Use is made there of characters – bounded, multiplicative or additive according to notation; the local inner product [Hey1, 5.1.7] is between the group and its Pontryagin dual.

One link between the group and vector-space aspects can be seen in the central role played in each by Gaussianity. We may think of this in each case as saying that, as in (CM), the relevant Fourier transform is of exponential type, the exponent having two terms, one linear (concerning means – location, or translation), one quadratic (concerning covariances, which captures scale and dependence effects). Where the density of the measure exists, it involves (via the 'normal' Edgeworth formula above in the Euclidean case) the inverse of the covariance Σ (matrix or operator), important in its own right (as the concentration or precision matrix/operator $K := \Sigma^{-1}$). So 'degeneracy-support' phenomena as above are unavoidable (below). Statistically, samples from two populations can only be usefully compared if their covariances are the same, and then the relevant statistic is the likelihood ratio; see e.g. [IbrR] for background here.

The supports of Gaussian measures on groups, and in particular the connections between Gaussian and Haar supports, have been studied by McCrudden [McC1,2], [McCW] and others.

13. *Dichotomy.* The equivalence-singularity dichotomy for Gaussian measures is a general consequence ([LePM], [Kal]; [MarR, §5.3]) of the triviality of a certain tail algebra (cf. [Hey1, Th. 5.5.6]); tail triviality in this case is established using a *zero-one law.*

14. *Automatic continuity.* The general theme of automatic continuity – situations where a function subject to mild qualitative conditions must necessarily be continuous – is important in many contexts; see e.g. [BinO5] and the references therein. For results of this type on γ-measurable linear functions for Gaussian γ, see [Bog1, Ch. 2]. See also [Sol, Cor. 2].

15. *Simmons-Mospan theorem and subcontinuity.* Recall from the companion paper [BinO7] (cf. [BinO4]) the following definition, already used in Th. 3.4 above. For $\mu \in \mathcal{P}(G)$ and $K \in \mathcal{K}(G)$,

$$\mu_{-}(K) := \sup_{\delta>0} \inf\{\mu(Kt) : t \in B_\delta\};$$

then μ is *subcontinuous* if $\mu_{-}(K) > 0$ for all K with $\mu(K) > 0$. (For a related notion see [LiuR], where a Radon measure μ on a space X, on which a group G acts homeomorphically, is called *mobile* if each map $t \mapsto \mu(Kt)$ is continuous for $K \in \mathcal{K}(X)$.) It follows from Prop. 3.1 above (see Cor. 3.2) that if $\mu_{-}(K) > 0$ for some K, then G is locally compact. Note that in a locally compact group, right uniform continuity of all the maps $t \mapsto \mu(tB)$ for B Borel is equivalent to absolute continuity of μ w.r.t. Haar measure ([Hey1, L. 6.3.4] – cf. [HewR, Th. 20.4]). So if G is not locally compact, no measure $\mu \in \mathcal{P}(G)$ is subcontinuous; then for all compact $K \subseteq G$, $\mu_{-}(K) = 0$.

If G is locally compact, then its left Haar measure $\eta = \eta_G$ satisfies

$$\eta_{-}(K) = \eta(K);$$

in particular, this equation holds for all non-null compact K. The latter observation extends to measures μ that are absolutely continuous w.r.t. η_G. Conversely, if μ is a measure satisfying $\mu_{-}(K) > 0$ for all compact K with $\mu(K) > 0$, then, as a consequence of the Simmons-Mospan theorem (§1), μ is absolutely continuous w.r.t. η_G : see [BinO7].

16. *Quasi-invariance and the Mackey topology of analytic Borel groups.* We stop to comment on the force of *full quasi-invariance* of a measure in connection with a Steinhaus triple (H, G, μ) with H (and G) Polish. Both groups, being absolutely Borel, are analytic spaces (Lemma 2.1 above). So both carry a *standard* Borel 'structure' (i.e. Borel isomorphic to the σ-algebra of Borel subsets of some Borel subset of a Polish space) with H carrying a Borel 'substructure' (σ-subalgebra) of G. (Borel subsets of H are Borel in G.) Mackey [Mac] investigates such Borel groups, defining also a (Borel) measure μ to be *standard* if it has a *standard* Borel support (vanishes outside of a standard Borel set). It emerges that every σ-finite Borel measure in an analytic Borel space is standard [Mac, Th. 6.1]. Of interest to us is Mackey's notion of a 'measure class' C_μ, comprising all Borel measures ν with the same null sets as $\mu : \mathcal{M}_0(\nu) = \mathcal{M}_0(\mu)$. Such a measure class may be closed under translation, and may be right or left invariant; then the common null sets are themselves invariant, and so may be viewed as witnessing quasi-invariance of the measure μ. Mackey shows that a Borel group with a one-sided invariant measure class has a both-sidedly invariant measure class [Mac, Lemma 7.2]; furthermore, if the class is countably generated, then the class contains a left-invariant and a right-invariant measure [Mac, Lemma 7.3]. This enables Mackey to improve on Weil's theorem in showing that an analytic Borel group G with a one-sidedly invariant measure class, in particular one generated by a quasi-invariant measure, has a unique locally compact topology making G a topological group as well as generating the given Borel 'structure'.

17. *The Strassen set and the law of the iterated logarithm (LIL).* The LIL completes (with the law of large numbers (LLN) and central limit theorem (CLT)) the trilogy of classical limit theorems in probability theory; for a survey see e.g. [Bin]. One form, the compact LIL, links the unit ball U of the reproducing-kernel Hilbert space associated with the covariance structure of a random variable X with values in a separable Banach space B with the cluster set of the partial sums, normalised as in the classical LIL. See e.g. [LedT, 207-210]. The first results of this type were Strassen's functional LIL and its extension to Banach spaces by Kuelbs and others ([LedT, 233-234], [Bog1, 358]).

18. *Product measures.* Infinite products of probability measures correspond to infinite sequences of independent random variables; they give a particularly important class of probability measures on infinite-dimensional spaces. A basic result here is the *Kakutani alternative*: if the laws of the factors are equivalent, the laws of the products are either equivalent or mutually singular, depending on the convergence or divergence of the infinite product of the Hellinger distances of the factor laws ([Kak2], [JacS, Ch. IV]; the term *Kakutani-Hellinger distance* is now used). As usual with dichotomies in probability theory, there are links with zero-one laws (cf. §9.13). See also Shepp [She], [Kak2].

19. *Non-Archimedean fields.* Löwner's result [Loe] (cf. [Neu2]) addresses the loss of a property – desirable, in some respects – as the dimension $n \to \infty$, by changing the base field from the reals to a non-Archimedean field. This is an early example of non-Archimedean fields (which originate in algebra and algebraic number theory) being applied to address a concrete problem in a quite different area.

20. *Other settings.* Recent generalizations of Cameron-Martin theory analyze an infinite-dimensional Lie-group or a sub-Riemannian manifold setting – see for example [DriG], [GorL] and [Gor], which thus preserve much of the classical setting; see also [Pug] (cf. [Bog1, p. 393]) and [Shi] for special cases.

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