Perpetual dual American barrier options for short sellers

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We obtain closed-form solutions to the problems of pricing of perpetual American put and call barrier options in the one-dimensional Black-Merton-Scholes model from the point of view of short sellers. The proof is based on the reduction of the original optimal stopping problems for a one-dimensional geometric Brownian motion with positive exponential discounting rates to the equivalent free-boundary problems and the solution of the latter problems by means of the smooth-fit conditions.

1 Introduction

The main aim of this paper is to present closed-form solutions to the optimal stopping problems of (2.3) for the geometric Brownian motion S defined in (2.1)-(2.2) with positive exponential discounting rates. The process S can describe the price of the underlying risky asset (e.g. a stock) in a model of a financial market. The values of (2.3) are then the rational (or no-arbitrage) prices of perpetual American barrier options in the Black-Merton-Scholes model from the point of view of short sellers (see, e.g. Shiryaev [27; Chapter VIII; Section 2a], Peskir and Shiryaev [22; Chapter VII; Section 25], or Detemple [10], for an extensive overview of other related results in the area).

Optimal stopping problems for one-dimensional diffusion processes with positive exponential discounting rates have been considered in Dynkin [12], Fakeev [13], Mucci [18], Salminen [25], Øksendal and Reikvam [20], and Beibel and Lerche [5]-[6] among others (see also Bensoussan and Lions [7; Theorem 3.19] and Øksendal [19; Chapter X]), for general rewards and infinite time horizon. More recently, such optimal stopping problems were studied in Dayanik and Karatzas [9], Alvarez [1]-[2], Peskir and Shiryaev [22], and Lamberton and Zervos [16] (see the latter references for an extensive discussion). Optimal stopping problems for one-dimensional continuous-time Markov processes with positive exponential discounting rates were recently considered by Shepp and Shiryaev [26], Xia and Zhou [28], Battauz et al. [3]-[4], and De Donno et al. [11] among others. The consideration of positive discounting rates implied the

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appearance of disconnected continuation regions or so-called double continuation regions. In the present paper, we derive explicit expressions for the value functions and stopping boundaries of some optimal stopping problems for one-dimensional geometric Brownian motions with positive exponential discounting rates. It is assumed that the rewards are equal to zero whenever the process reaches certain constant upper or lower levels, so that the value functions are equal to the rational values of perpetual dual American barrier options.

The paper is organised as follows. In Section 2, we introduce the setting and notations of the perpetual dual American up-and-out put and down-and-out call option pricing problems as optimal stopping problems for a geometric Brownian motion with a positive exponential discounting rate and formulate the associated free-boundary problems. In Section 3, we derive closed-form solutions of the latter problems under various relations between the parameters of the model. In Section 4, we verify that the solutions of the free-boundary problems provide the solutions of the original optimal stopping problems. The main results of the paper are stated in Propositions 1 and 2.

2 Preliminaries

In this section, we give a formulation of optimal stopping problems with positive exponential discounting rates related to the pricing of perpetual American barrier options from the point of view of short sellers.

2.1 The model

For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ and its natural filtration $(\mathcal{F}_t)_{t \geq 0}$. It is further assumed that the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and completed by all the sets of P-measure zero. Let us define the process $S = (S_t)_{t \geq 0}$ by

$$S_t = s \, \exp\Big(\big(r - \delta - \sigma^2/2 \big) \, t + \sigma \, B_t \Big) \tag{2.1}$$

which solves the stochastic differential equation

$$dS_t = (r - \delta) \, S_t \, dt + \sigma \, S_t \, dB_t \quad (S_0 = s) \tag{2.2}$$

where $s > 0$ is fixed, and $r > 0$, $\delta > 0$, and $\sigma > 0$ are some given constants. It is assumed that the process S describes the price of a risky asset on a financial market, where r is the riskless interest rate of a bank account, δ is the dividend rate paid to the asset holders, and σ is the volatility rate. The purpose of the present paper is to study the optimal stopping problems for the value functions

$$V_i^*(s) = \inf_{\tau_i} E_s \big[e^{r \tau_i} \, G_i(S_{\tau_i}) \, I(\tau_i < \zeta_i) \big] \tag{2.3}$$

with $G_1(s) = K_1 - s$ and $G_2(s) = s - K_2$, for some $K_i > 0$ fixed, where the infima are taken over all stopping times τ_i, $i = 1, 2$, with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Here E_s denotes the expectation with respect to the probability measure P under the assumption that the process

S starts at $s > 0$, and $I(\cdot)$ is the indicator function. We assume that the random times ζ_i, $i = 1, 2$, are given by

$$\zeta_1 = \inf\{t \geq 0 \,|\, S_t \geq b'\} \quad \text{and} \quad \zeta_2 = \inf\{t \geq 0 \,|\, S_t \leq a'\} \tag{2.4}$$

for some $0 < b' < K_1$ and $0 < K_2 < a'$ fixed. Since the initial probability measure P is a martingale martingale measure (see, e.g. [27; Chapter VII, Section 3g]), the values of (2.3) provide the rational (or no-arbitrage) prices of the perpetual American dual barrier down-and-out put and up-and-out call options, respectively. The operations of such contracts can be described as follows. It is assumed that the short sellers receive the fixed payments $V_i^*(s)$ at time 0 and incur obligations to deliver to the buyers the payoffs $e^{r\tau_i} G_i(S_{\tau_i}) I(\tau_i < \zeta_i)$ at some future times τ_i, $i = 1, 2$, which the sellers can choose. Observe that when $2r - \delta \leq 0$ holds, the process $(e^{rt} S_t)_{t \geq 0}$ is a supermartingale closed at zero, so that the optimal exercise time τ_1^* is zero, while the optimal exercise time τ_2^* coincides with ζ_2. In this respect, we further assume that $2r - \delta > 0$ holds.

2.2 The optimal exercise times

By means of the results of general theory of optimal stopping (see, e.g. [22; Chapter I, Section 2]), it follows from the structure of the rewards in (2.3) that the optimal stopping times in these problems are given by

$$\tau_i^* = \inf\{t \geq 0 \,|\, V_i^*(S_t) = G_i(S_t)\} \tag{2.5}$$

for every $i = 1, 2$. We further assume that the optimal stopping times in the problems of (2.3) are of the form

$$\tau_1^* = \inf\{t \geq 0 \,|\, S_t \notin (a_*, b')\} \quad \text{and} \quad \tau_2^* = \inf\{t \geq 0 \,|\, S_t \notin (a', b_*)\} \tag{2.6}$$

for some numbers $0 < a_* < b'$ and $0 < a' < b_*$ to be determined. By a standard application of Itô's formula (see, e.g. [17; Theorem 4.4]) to the process $e^{r(\tau_i^* \wedge t)} G_i(S_{\tau_i^* \wedge t})$, we obtain the representations

$$\begin{aligned} &e^{r(\tau_i^* \wedge t)} \, G_i(S_{\tau_i^* \wedge t}) \, I(t < \zeta_i) = G_i(s) \\ &+ (-1)^i \int_0^{\tau_i^* \wedge t} e^{ru} \, \big((2r - \delta) \, S_u - r \, K_i \big) \, I(u < \zeta_i) \, du + N_t^i \end{aligned} \tag{2.7}$$

for $s < b'$ or $s > a'$, where the process $(N^i_{\tau_i^* \wedge t})_{t \geq 0}$ defined by

$$N^i_{\tau_i^* \wedge t} = (-1)^i \int_0^{\tau_i^* \wedge t} e^{ru} \, I(u < \zeta_i) \, \sigma \, S_u \, dB_u \tag{2.8}$$

is a continuous square integrable martingale under the probability measure P_s, for every $i = 1, 2$. Hence, by applying Doob's optional sampling theorem (see, e.g. [17; Chapter III, Theorem 3.6] or [23; Chapter II, Theorem 3.2]), we obtain that the value functions in (2.3) admit the representations

$$V_i^*(s) = G_i(s) + (-1)^i \, E_s \bigg[\int_0^{\tau_i^*} e^{rt} \, \big((2r - \delta) \, S_t - r \, K_i \big) \, I(t < \zeta_i) \, dt \bigg] \tag{2.9}$$

for all $s < b'$ or $s > a'$, and every $i = 1, 2$. Thus, it is seen from the structure of the integrand in (2.9) that it is not optimal to exercise the barrier put and call options when $S_t > \overline{a}$ with $\overline{a} = rK_1/(2r - \delta)$ and $S_t < \underline{b}$ with $\underline{b} = rK_2/(2r - \delta)$, for any $0 \le t < \tau_i^* \wedge \zeta_i$, $i = 1, 2$, respectively. In this respect, we further assume that the optimal stopping boundaries a_* and b_* in (2.6) should satisfy the inequalities $a_* < \overline{a}$ and $b_* > \underline{b}$, respectively.

2.3 The free-boundary problems

It can be shown by means of Itô's formula that the infinitesimal operator $\mathbb{L}$ of the process S acts on a locally bounded twice continuously differentiable function $F(s)$ on $(0, \infty)$ in the form

$$(\mathbb{L}F)(s) = (r - \delta)\, s\, F'(s) + \frac{\sigma^2 s^2}{2}\, F''(s) \tag{2.10}$$

for all $s > 0$. In order to find closed-form expressions for the unknown value functions $V_i^*(s)$, $i = 1, 2$, from (2.3) and the unknown boundaries a_* and b_* from (2.6), we may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [22; Chapter IV, Section 8]) and formulate the associated free-boundary problems

$$(\mathbb{L}V_i)(s) = -rV_i(s) \quad \text{for } a < s < b' \text{ or } a' < s < b \text{ and } i = 1, 2 \tag{2.11}$$
$$V_1(s)\big|_{s=a+} = K_1 - a, \quad V_2(s)\big|_{s=b-} = b - K_2 \tag{2.12}$$
$$V_1'(s)\big|_{s=a+} = -1, \quad V_2(s)\big|_{s=b-} = 1 \tag{2.13}$$
$$V_2'(s)\big|_{s=b'-} = 0, \quad V_2(s)\big|_{s=a'+} = 0 \tag{2.14}$$
$$V_1(s) = K_1 - s \;\text{ for }\; s < a, \quad V_2(s) = s - K_2 \;\text{ for }\; s > b \tag{2.15}$$
$$V_1(s) < K_1 - s \;\text{ for }\; a < s < b', \quad V_2(s) < s - K_2 \;\text{ for }\; a' < s < b \tag{2.16}$$
$$(\mathbb{L}V_i)(s) > -rV_i(s) \quad \text{for }\; s < a \;\text{ or }\; s > b \;\text{ and }\; i = 1, 2 \tag{2.17}$$

for some $0 < a < b' < K_1$ and $0 < K_2 < a' < b$ to be determined. Observe that the superharmonic characterisation of the value function (see, e.g. [22; Chapter IV, Section 9]) implies that $V_i^*(s)$, $i = 1, 2$, are the smallest functions satisfying (2.11)-(2.12) and (2.15)-(2.16) with the boundaries a_* and b_*, respectively.

3 Solutions to the free-boundary problems

We now look for functions which solve the free-boundary problems stated in (2.11)-(2.17). For this purpose, we consider three separate cases based on the different relations between the parameters of the model (see Figures 1 and 2 below for computer drawings of the value functions $V_i^*(s)$, $i = 1, 2$).

3.1 The case $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$

Let us first assume that $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds. Then, the general solution of the second-order ordinary differential equation in (2.11) has the form

$$V_i(s) = C_{i,1}\, s^{\eta_1} + C_{i,2}\, s^{\eta_2} \tag{3.1}$$

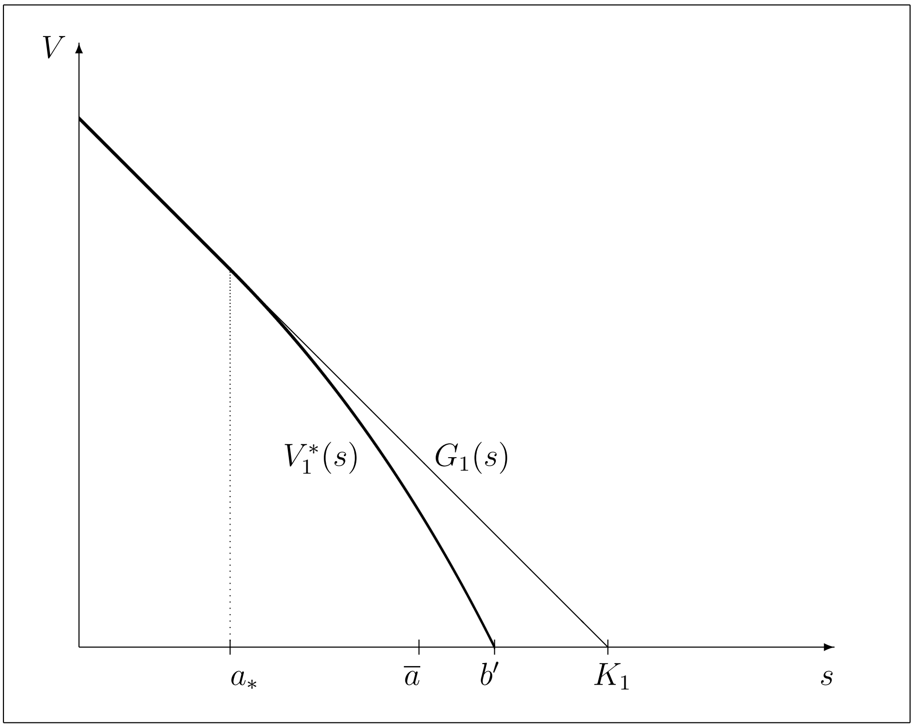

Figure 1. A computer drawing of the value function $V_1^*(s)$ of the put option.

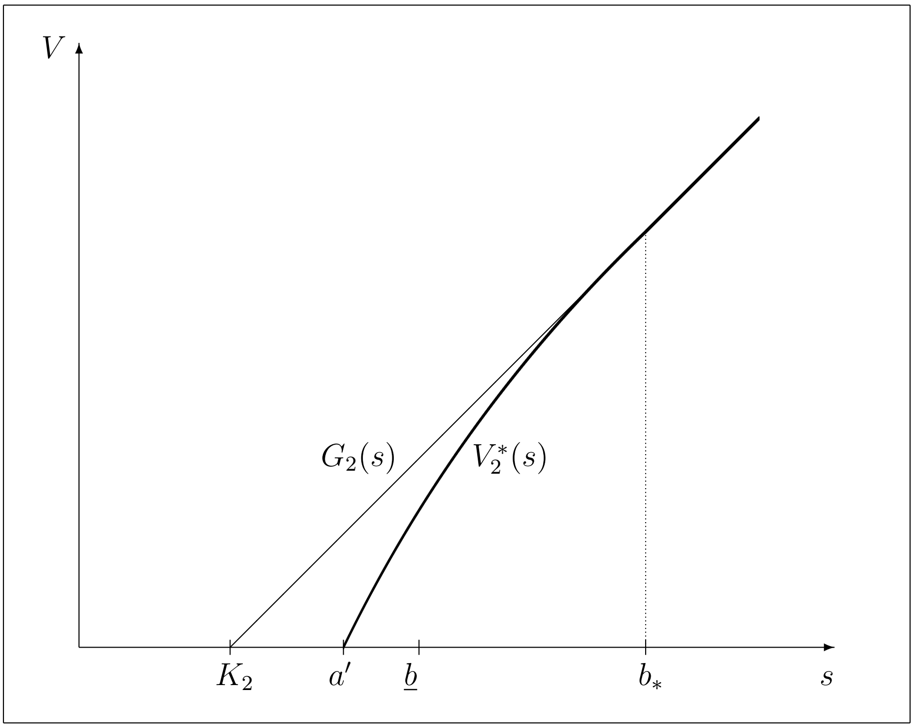

Figure 2. A computer drawing of the value function $V_2^*(s)$ of the call option.

where $C_{i,j}$, $i,j=1,2$, are some arbitrary constants, and η_j, $j=1,2$, are given by

$$\eta_j = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 - \frac{2r}{\sigma^2}} \tag{3.2}$$

so that the identity

$$\frac{\eta_1}{\eta_1 - 1} \, \frac{\eta_2}{\eta_2 - 1} = \frac{r}{2r-\delta} \tag{3.3}$$

is satisfied. Note that when $r-\delta < -\sigma^2/2$ holds, we have $1 < \eta_2 < \eta_1$, so that $1 < \eta_1/(\eta_1 - 1) < \eta_2/(\eta_2 - 1) < r/(2r-\delta)$ and $(\eta_1 - 1)/(\eta_2 - 1) > 1$. Moreover, when $-\sigma^2/2 \le r-\delta < \sigma^2/2$ holds, we have $0 < \eta_2 < \eta_1 < 1$, so that $\eta_1/(\eta_1 - 1) < \eta_2/(\eta_2 - 1) < 0$ and $0 < (\eta_1 - 1)/(\eta_2 - 1) < 1$. Finally, when $r - \delta > \sigma^2/2$ holds, we have $\eta_2 < \eta_1 < 0$, so that $0 < r/(2r-\delta) < \eta_1/(\eta_1 - 1) < \eta_2/(\eta_2 - 1) < 1$ and $0 < (\eta_1 - 1)/(\eta_2 - 1) < 1$.

Then, by applying the conditions from (2.12)-(2.14) to the function in (3.1), we get that the equalities

$$C_{1,1}\, a^{\eta_1} + C_{1,2}\, a^{\eta_2} = K_1 - a, \quad C_{2,1}\, b^{\eta_1} + C_{2,2}\, b^{\eta_2} = b - K_2 \tag{3.4}$$

$$C_{1,1}\, \eta_1\, a^{\eta_1} + C_{1,2}\, \eta_2\, a^{\eta_2} = -a, \quad C_{2,1}\, \eta_1\, b^{\eta_1} + C_{2,2}\, \eta_2\, b^{\eta_2} = b \tag{3.5}$$

$$C_{1,1}\, (b')^{\eta_1} + C_{1,2}\, (b')^{\eta_2} = 0, \quad C_{2,1}\, (a')^{\eta_1} + C_{2,2}\, (a')^{\eta_2} \tag{3.6}$$

should hold for some $0 < a < b' < K_1$ and $0 < K_2 < a' < b$. Hence, solving the systems in (3.4)-(3.6), we obtain that the candidate value function has the form

$$\begin{aligned} & V_1(s; a_*, b') \\ & = \frac{1}{\eta_1 - \eta_2} \left(\big((\eta_2 - 1)\, a_* - \eta_2\, K_1 \big) \left(\frac{s}{a_*}\right)^{\eta_1} + \big((1 - \eta_1)\, a_* + \eta_1\, K_1 \big) \left(\frac{s}{a_*}\right)^{\eta_2} \right) \end{aligned} \tag{3.7}$$

for $a_* < s < b' < K_1$, and

$$\begin{aligned} & V_2(s; b_*, a') \\ & = \frac{1}{\eta_1 - \eta_2} \left(\big((1 - \eta_2)\, b_* + \eta_2\, K_2 \big) \left(\frac{s}{b_*}\right)^{\eta_1} + \big((\eta_1 - 1)\, b_* - \eta_1\, K_2 \big) \left(\frac{s}{b_*}\right)^{\eta_2} \right) \end{aligned} \tag{3.8}$$

for $K_2 < a' < s < b_*$, where a_* and b_* are determined from the arithmetic equations

$$\frac{\eta_1 K_1 - (\eta_1 - 1) a}{\eta_2 K_1 - (\eta_2 - 1) a} \equiv \frac{\eta_1 - 1}{\eta_2 - 1} + \frac{(\eta_1 - \eta_2) K_1}{(\eta_2 - 1)^2} \left(a - \frac{\eta_2 K_1}{\eta_2 - 1} \right)^{-1} = \left(\frac{b'}{a} \right)^{\eta_1 - \eta_2} \tag{3.9}$$

and

$$\frac{(\eta_1 - 1) b - \eta_1 K_2}{(\eta_2 - 1) b - \eta_2 K_2} \equiv \frac{\eta_1 - 1}{\eta_2 - 1} + \frac{(\eta_1 - \eta_2) K_2}{(\eta_2 - 1)^2} \left(b - \frac{\eta_2 K_2}{\eta_2 - 1} \right)^{-1} = \left(\frac{a'}{b} \right)^{\eta_1 - \eta_2} \tag{3.10}$$

respectively.

In order to consider the put option case, we observe from the mentioned above properties of the numbers η_j, $j=1,2$, from (3.2) and the identity in (3.3) that, if $r-\delta < -\sigma^2/2$ holds, then the equation in (3.9) has a unique solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$ with

$\overline{a} = rK_1/(2r-\delta)$. Then, if $-\sigma^2/2 \le r-\delta < \sigma^2/2$ holds, then the equation in (3.9) has a unique solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$, whenever either the inequality $r \le \delta$ or the inequalities $r > \delta$ and $b' \le \overline{a}$ are satisfied. Finally, if $r - \delta > \sigma^2/2$ holds, then the equation in (3.9) has no solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$, and thus, we can set $a_* = 0$, so that $V_1(s; a_*, b') = 0$, for all $0 < s < b'$.

In order to consider the call option case, we observe from the mentioned properties of η_j, $j = 1, 2$, and the identity above that, if $r - \delta > \sigma^2/2$ holds, then the equation in (3.10) has a unique solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$ with $\underline{b} = rK_2/(2r-\delta)$. Then, if $-\sigma^2/2 \le r-\delta < \sigma^2/2$ holds, then the equation in (3.10) has a unique solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$, whenever either the inequality $r \ge \delta$ or the inequalities $r < \delta$ and $a' \ge \underline{b}$ are satisfied. Finally, if $r - \delta < -\sigma^2/2$ holds, then the equation in (3.10) has no solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$, and thus, we can set $b_* = \infty$, so that $V_2(s; b_*, a') = 0$, for all $s > a'$.

3.2 The case $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$

Let us now assume that $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds. Then, the general solution of the ordinary differential equation in (2.11) has the form

$$V_i(s) = C_{i,1}\, s^{\lambda} \ln s + C_{i,2}\, s^{\lambda} \tag{3.11}$$

where $C_{i,j}$, $i, j = 1, 2$, are some arbitrary constants, and λ is given by:

$$\lambda = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \tag{3.12}$$

so that the identity

$$\left(\frac{\lambda}{\lambda-1}\right)^2 = \frac{r}{2r-\delta} \tag{3.13}$$

is satisfied. Observe that the value of λ in (3.12) coincides with the values of η_i, $i = 1, 2$, under the current assumption $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$, since the appropriate expression under the root sign becomes zero in this case. We also note that when $r - \delta < -\sigma^2/2$ holds, we have $\lambda > 1$, so that $\lambda/(\lambda-1) > 1$ and $1/(\lambda-1) > 0$. Moreover, when $-\sigma^2/2 \le r-\delta < \sigma^2/2$ holds, we have $0 < \lambda < 1$, so that $\lambda/(\lambda-1) < 0$ and $1/(\lambda-1) < 0$. Finally, when $r - \delta > \sigma^2/2$ holds, we have $\lambda < 0$, so that $0 < \lambda/(\lambda-1) < 1$ and $1/(\lambda-1) < 0$.

Then, by applying the conditions from (2.12)-(2.14) to the function in (3.11), we get that the equalities

$$C_{1,1}\, a^{\lambda} \ln a + C_{1,2}\, a^{\lambda} = K_1 - a, \quad C_{2,1}\, b^{\lambda} \ln b + C_{2,2}\, b^{\lambda} = b - K_2 \tag{3.14}$$

$$C_{1,1}\, a^{\lambda} (\lambda \ln a + 1) + C_{1,2}\, \lambda\, a^{\lambda} = -a, \quad C_{2,1}\, b^{\lambda} (\lambda \ln b + 1) + C_{2,2}\, \lambda\, b^{\lambda} = b \tag{3.15}$$

$$C_{1,1}\, (b')^{\lambda} \ln b' + C_{1,2}\, (b')^{\lambda} = 0, \quad C_{2,1}\, (a')^{\lambda} \ln a' + C_{2,2}\, (a')^{\lambda} = 0 \tag{3.16}$$

should hold for some $0 < a < b' < K_1$ and $0 < K_2 < a' < b$. Thus, solving the systems in (3.14)-(3.16), we obtain that the candidate value function has the form

$$V_1(s; a_*, b') = \big((\lambda - 1)\, a_* - \lambda\, K_1\big) \left(\frac{s}{a_*}\right)^{\lambda} \ln\left(\frac{s}{a_*}\right) + (K_1 - a_*) \left(\frac{s}{a_*}\right)^{\lambda} \tag{3.17}$$

for $a_* < s < b' < K_1$, and

$$V_2(s; b_*, a') = \big(\lambda\, K_2 - (\lambda - 1)\, b_*\big) \left(\frac{s}{b_*}\right)^{\lambda} \ln\left(\frac{s}{b_*}\right) + (b_* - K_2) \left(\frac{s}{b_*}\right)^{\lambda} \tag{3.18}$$

for $K_2 < a' < s < b_*$, where a_* and b_* are determined from the arithmetic equations

$$\frac{K_1 - a}{\lambda K_1 - (\lambda - 1)a} \equiv \frac{1}{\lambda - 1} + \frac{K_1}{(\lambda - 1)^2} \left(a - \frac{\lambda K_1}{\lambda - 1}\right)^{-1} = \ln\left(\frac{b'}{a}\right) \tag{3.19}$$

and

$$\frac{b - K_2}{(\lambda - 1)b - \lambda K_2} \equiv \frac{1}{\lambda - 1} + \frac{K_2}{(\lambda - 1)^2} \left(b - \frac{\lambda K_2}{\lambda - 1}\right)^{-1} = \ln\left(\frac{a'}{b}\right) \tag{3.20}$$

respectively.

In order to consider the put option case, we observe from the expressions for λ in (3.12) and (3.13) that, if $r - \delta < -\sigma^2/2$ holds, then the equation in (3.19) has a unique solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$ with $\overline{a} = rK_1/(2r - \delta)$. Then, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ holds, then the equation in (3.19) has a unique solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$, whenever either the inequality $r \le \delta$ or the inequalities $r > \delta$ and $b' \le \overline{a}$ are satisfied. Finally, if $r - \delta > \sigma^2/2$ holds, then the equation in (3.19) has no solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$, and thus, we can set $a_* = 0$, so that $V_1(s; a_*, b') = 0$, for all $0 < s < b'$.

In order to consider the call option case, we observe from the mentioned above properties of λ that, if $r - \delta > \sigma^2/2$ holds, then the equation in (3.20) has a unique solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$ with $\underline{b} = rK_2/(2r - \delta)$. Then, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ holds, then the equation in (3.20) has a unique solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$, whenever either the inequality $r \ge \delta$ or the inequalities $r < \delta$ and $a' \ge \underline{b}$ are satisfied. Finally, if $r - \delta < -\sigma^2/2$ holds, then the equation in (3.20) has no solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$, and thus, we can set $b_* = \infty$, so that $V_2(s; b_*, a') = 0$, for all $s > a'$.

3.3 The case $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$

Let us finally assume that $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds. Then, the general solution of the ordinary differential equation in (2.11) has the form

$$V_i(s) = C_{i,1}\, s^{\lambda} \sin\big(\theta \ln s\big) + C_{i,2}\, s^{\lambda} \cos\big(\theta \ln s\big) \tag{3.21}$$

where $C_{i,j}$, $i, j = 1, 2$, are some arbitrary constants, while λ is given by (3.12) and θ is set as

$$\theta = \sqrt{\frac{2r}{\sigma^2} - \left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2}. \tag{3.22}$$

Hence, by applying the conditions from (2.12)-(2.14) to the function in (3.21), we get that the equalities

$$C_{1,1}\, a^{\lambda} \sin\big(\theta \ln a\big) + C_{1,2}\, a^{\lambda} \cos\big(\theta \ln a\big) = K_1 - a, \tag{3.23}$$

$$C_{2,1}\, b^{\lambda} \sin\big(\theta \ln b\big) + C_{2,2}\, b^{\lambda} \cos\big(\theta \ln b\big) = b - K_2 \tag{3.24}$$

$$(C_{1,1}\, \lambda - C_{1,2}\, \theta)\, a^{\lambda} \sin\big(\theta \ln a\big) + (C_{1,1}\, \theta + C_{1,2}\, \lambda)\, a^{\lambda} \cos\big(\theta \ln a\big) = -a, \tag{3.25}$$

$$(C_{2,1}\, \lambda - C_{2,2}\, \theta)\, a^{\lambda} \sin\big(\theta \ln a\big) + (C_{2,1}\, \theta + C_{2,2}\, \lambda)\, a^{\lambda} \cos\big(\theta \ln a\big) = b \tag{3.26}$$

$$C_{1,1}\, (b')^{\lambda} \sin\big(\theta \ln b'\big) + C_{1,2}\, (b')^{\lambda} \cos\big(\theta \ln b'\big) = 0, \tag{3.27}$$

$$C_{2,1}\, (a')^{\lambda} \sin\big(\theta \ln a'\big) + C_{2,2}\, (a')^{\lambda} \cos\big(\theta \ln a'\big) = 0 \tag{3.28}$$

should hold for some $0 < a < b' < K_1$ and $0 < K_2 < a' < b$. Thus, solving the systems in (3.23)-(3.28), we obtain that the candidate value function has the form

$$\begin{aligned} &V_1(s; a_*, b') \\ &= \big((\lambda-1)a_* - \lambda K_1\big) \left(\frac{s}{a_*}\right)^{\lambda} \sin\left(\theta \ln\left(\frac{s}{a_*}\right)\right) + \theta(K_1 - a_*) \left(\frac{s}{a_*}\right)^{\lambda} \cos\left(\theta \ln\left(\frac{s}{a_*}\right)\right) \end{aligned} \tag{3.29}$$

for $a_* < s < b' < K_1$, and

$$\begin{aligned} &V_2(s; b_*, a') \\ &= \big(\lambda K_2 - (\lambda-1)b_*\big) \left(\frac{s}{b_*}\right)^{\lambda} \sin\left(\theta \ln\left(\frac{s}{b_*}\right)\right) + \theta(b_* - K_2) \left(\frac{s}{b_*}\right)^{\lambda} \cos\left(\theta \ln\left(\frac{s}{b_*}\right)\right) \end{aligned} \tag{3.30}$$

for $K_2 < a' < s < b_*$, where a_* and b_* are determined from the arithmetic equations

$$\arctan\left(\frac{\theta}{\lambda-1} + \frac{\theta K_1}{(\lambda-1)^2}\left(a - \frac{\lambda K_1}{\lambda-1}\right)^{-1}\right) = \theta \ln\left(\frac{b'}{a}\right) \tag{3.31}$$

and

$$\arctan\left(\frac{\theta}{\lambda-1} + \frac{\theta K_2}{(\lambda-1)^2}\left(b - \frac{\lambda K_2}{\lambda-1}\right)^{-1}\right) = \theta \ln\left(\frac{a'}{b}\right) \tag{3.32}$$

respectively.

In order to consider the put option case, we observe from the expressions for λ in (3.12) and (3.13) that, if $r - \delta < -\sigma^2/2$ holds, then the equation in (3.31) has a unique solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$ with $\overline{a} = rK_1/(2r-\delta)$. Then, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ holds, then the equation in (3.31) has a unique solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$, whenever either the inequality $r \le \delta$ or the inequalities $r > \delta$ and $b' \le \overline{a}$ are satisfied. Finally, if $r - \delta > \sigma^2/2$ holds, then the equation in (3.31) has no solution a_* on the interval $(0, b')$ such that $a_* < \overline{a}$, and thus, we can set $a_* = 0$, so that $V_1(s; a_*, b') = 0$, for all $0 < s < b'$.

In order to consider the call option case, we observe from the mentioned above properties of λ that, if $r - \delta > \sigma^2/2$ holds, then the equation in (3.32) has a unique solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$ with $\underline{b} = rK_2/(2r-\delta)$. Then, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ holds, then the equation in (3.32) has a unique solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$, whenever either the inequality $r \ge \delta$ or the inequalities $r < \delta$ and $a' \ge \underline{b}$ are satisfied. Finally, if $r - \delta < -\sigma^2/2$ holds, then the equation in (3.32) has no solution b_* on the interval (a', ∞) such that $b_* > \underline{b}$, and thus, we can set $b_* = \infty$, so that $V_2(s; b_*, a') = 0$, for all $s > a'$.

4 Main results

In this section, we show that the solutions of the free-boundary problems from the previous section provides the solutions of the initial optimal stopping problems of (2.3).

Proposition 4.1 *Let the process S be given by (2.1), with some $r > 0$, $\delta > 0$, and $\sigma > 0$ fixed, and such that $2r - \delta > 0$. Then, the value function of the perpetual American dual barrier (up-and-out) put option in (2.3) has the form*

$$V_1^*(s) = \begin{cases} V_1(s; a_*, b'), & \text{if } a_* < s < b' \\ K_1 - s, & \text{if } s \le a_* \end{cases} \tag{4.1}$$

for some $0 < b' < K_1$ fixed, and τ_1^* from (2.6) is an optimal stopping time, where we have the following assertions:

(i) When $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, the function $V_1(s; a_*, b')$ takes the expression of (3.7), while if $r - \delta < -\sigma^2/2$ then the equation in (3.9) has a unique solution a_* on the interval $(0, b' \wedge \overline{a})$ with $\overline{a} = rK_1/(2r - \delta)$, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ and either the inequality $r \le \delta$ or the inequalities $r > \delta$ and $b' \le \overline{a}$ are satisfied then the equation in (3.9) has a unique solution a_* on the interval $(0, b' \wedge \overline{a})$, as well as if $r - \delta > \sigma^2/2$ then the equation in (3.9) has no solution a_* on the interval $(0, b' \wedge \overline{a})$, so that $a_* = 0$ and $V_1(s; a_*, b') \equiv 0$.

(ii) When $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, the function $V_1(s; a_*, b')$ takes the expression of (3.17), while if $r - \delta < -\sigma^2/2$ then the equation in (3.19) has a unique solution a_* on the interval $(0, b' \wedge \overline{a})$, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ and either the inequality $r \le \delta$ or the inequalities $r > \delta$ and $b' \le \overline{a}$ are satisfied then the equation in (3.19) has a unique solution a_* on the interval $(0, b' \wedge \overline{a})$, as well as if $r - \delta > \sigma^2/2$ then the equation in (3.19) has no solution a_* on the interval $(0, b' \wedge \overline{a})$, so that $a_* = 0$ and $V_1(s; a_*, b') \equiv 0$.

(iii) When $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, the function $V_1(s; a_*, b')$ takes the expression of (3.29), while if $r - \delta < -\sigma^2/2$ then the equation in (3.31) has a unique solution a_* on the interval $(0, b' \wedge \overline{a})$, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ and either the inequality $r \le \delta$ or the inequalities $r > \delta$ and $b' \le \overline{a}$ are satisfied then the equation in (3.31) has a unique solution a_* on the interval $(0, b' \wedge \overline{a})$, as well as if $r - \delta > \sigma^2/2$ then the equation in (3.31) has no solution a_* on the interval $(0, b' \wedge \overline{a})$, so that $a_* = 0$ and $V_1(s; a_*, b') \equiv 0$.

Proposition 4.2 *Let the process S be given by (2.1), with some $r > 0$, $\delta > 0$, and $\sigma > 0$ fixed, and such that $2r - \delta > 0$. Then, the value function of the perpetual American dual barrier (down-and-out) call option in (2.3) has the form:*

$$V_2^*(s) = \begin{cases} V_2(s; b_*, a'), & \text{if } a' < s < b_* \\ s - K_2, & \text{if } s \ge b_* \end{cases} \tag{4.2}$$

for some $0 < K_2 < a'$ fixed, and τ_2^* from (2.6) is an optimal stopping time, where we have the following assertions:

(i) When $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, the function $V_2(s; b_*, a')$ takes the expression of (3.8), while if $r - \delta > \sigma^2/2$ then the equation in (3.10) has a unique solution b_* on the interval $(a' \vee \underline{b}, \infty)$ with $\underline{b} = rK_2/(2r - \delta)$, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ and either the inequality $r \ge \delta$ or the inequalities $r < \delta$ and $a' \ge \underline{b}$ are satisfied then the equation in (3.10) has a unique solution

b_* on the interval $(a' \vee \underline{b}, \infty)$, while if $r - \delta < -\sigma^2/2$ then the equation in (3.10) has no solution b_* on the interval $(a' \vee \underline{b}, \infty)$, so that $b_* = \infty$ and $V_2(s; b_*, a') \equiv 0$.

(ii) When $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, the function $V_2(s; b_*, a')$ takes the expression of (3.18), while if $r - \delta > \sigma^2/2$ then the equation in (3.20) has a unique solution b_* on the interval $(a' \vee \underline{b}, \infty)$, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ and either the inequality $r \ge \delta$ or the inequalities $r < \delta$ and $a' \ge \underline{b}$ are satisfied then the equation in (3.20) has a unique solution b_* on the interval $(a' \vee \underline{b}, \infty)$, as well as if $r - \delta < -\sigma^2/2$ then the equation in (3.20) has no solution b_* on the interval $(a' \vee \underline{b}, \infty)$, so that $b_* = \infty$ and $V_2(s; b_*, a') \equiv 0$.

(iii) When $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ holds, the function $V_2(s; b_*, a')$ takes the expression of (3.30), while if $r - \delta > \sigma^2/2$ then the equation in (3.32) has a unique solution b_* on the interval $(a' \vee \underline{b}, \infty)$, if $-\sigma^2/2 \le r - \delta < \sigma^2/2$ and either the inequality $r \ge \delta$ or the inequalities $r < \delta$ and $a' \ge \underline{b}$ are satisfied then the equation in (3.32) has a unique solution b_* on the interval $(a' \vee \underline{b}, \infty)$, as well as if $r - \delta < -\sigma^2/2$ then the equation in (3.32) has no solution b_* on the interval $(a' \vee \underline{b}, \infty)$, so that $b_* = \infty$ and $V_2(s; b_*, a') \equiv 0$.

Proof: In order to verify the assertions stated above, we are left to show that the functions introduced in (4.1) and (4.2) coincide with the value functions in (2.3), and that the stopping times τ_i^*, $i = 1, 2$, in (2.6) are optimal with the boundaries a_* and b_* specified above. For this purpose, let us denote by $V_i(s)$, $i = 1, 2$, the right-hand sides of the expressions in (4.1) and (4.2). Then, we may conclude from the equations in (2.11) that the derivatives $V_i'(s)$, $i = 1, 2$, are continuously differentiable on (a_*, b') and (a', b_*), respectively. Hence, according to the conditions of (2.12)-(2.15), applying the change-of-variable formula from [21] (see also [22; Chapter II, Section 3.5] for a summary of the related results on the local time-space formula as well as further references), we get

$$e^{rt}\, V_i(S_t) = V_i(s) + \int_0^t e^{ru}\, (\mathbb{L}V_i + rV_i)(S_u) I(S_u \neq a_* \text{ or } S_u \neq b_*)\, du + M_t^i \tag{4.3}$$

for all $t \ge 0$, where the processes $M^i = (M_t^i)_{t \ge 0}$, $i = 1, 2$, defined by:

$$M_t^i = \int_0^t e^{ru}\, V_i'(S_u)\, \sigma\, S_u\, dB_u \tag{4.4}$$

are continuous local martingales with respect to the probability measure P_s. Observe that the time spent by S at the points a_* and b_* is of Lebesgue measure zero, and thus, the indicator which appear in the integral of (4.3) can be ignored (see, e.g. [8; Chapter II, Section 1]).

By using straightforward calculations and the arguments from the previous section, it is verified that the inequalities $(\mathbb{L}V_i + rV_i)(s) \ge 0$, $i = 1, 2$, hold, for all $s < b'$ such that $s \neq a_*$ or $s > a'$ such that $s \neq b_*$, respectively. Moreover, it is shown by means of standard arguments that the inequalities in (2.16) hold, which together with the conditions of (2.12)-(2.15) imply that $V_i(s) \le G_i(s)$, $i = 1, 2$, holds, for all $s < b'$ or $s > a'$, respectively. Hence, the expression in (4.3) yields that the inequalities

$$e^{r\tau_i}\, G_i(S_{\tau_i}) \ge e^{r\tau_i}\, V_i(S_{\tau_i}) \ge V_i(s) + M_{\tau_i}^i \tag{4.5}$$

hold for any stopping times τ_i, $i = 1, 2$, of the process S started at $s > 0$. Let $(\varkappa_i^n)_{n \in \mathbb{N}}$ be arbitrary localising sequences of stopping times for the processes M^i, $i = 1, 2$, respectively.

Taking in (4.5) the expectation with respect to the measure P_s, by means of the optional sampling theorem (see, e.g. [15; Chapter I, Theorem 3.22]), we get that the inequalities

$$E_s\big[e^{r(\tau_i \wedge \varkappa_i^n)}\, G_i(S_{\tau_i \wedge \varkappa_i^n})\big] \ge E_s\big[e^{r(\tau_i \wedge \varkappa_i^n)}\, V_i(S_{\tau_i \wedge \varkappa_i^n})\big] \tag{4.6}$$
$$\ge V_i(s) + E_s\big[M^i_{\tau_i \wedge \varkappa_i^n}\big] = V_i(s)$$

hold, for all $s > 0$ and every $i = 1, 2$. Hence, letting n go to infinity and using Fatou's lemma, we obtain

$$E_s\big[e^{r\tau_i}\, G(S_{\tau_i})\big] \ge E_s\big[e^{r\tau_i}\, V(S_{\tau_i})\big] \ge V_i(s) \tag{4.7}$$

for any stopping times τ_i, $i = 1, 2$, and all $s > 0$. By virtue of the structure of the stopping times in (2.6), it is readily seen that the equalities in (4.7) hold with τ_i^* instead of τ_i, $i = 1, 2$, when either $s \le a_*$ or $s \ge b_*$.

It remains us to show that the equalities are attained in (4.7) when τ_i^* replace τ_i, $i = 1, 2$, for $a_* < s < b'$ or $a' < s < b_*$, respectively. By virtue of the fact that the functions $V_1(s; a_*, b')$ and $V_2(s; b_*, a')$ and the boundaries a_* and b_* satisfy the conditions in (2.11) and (2.12), it follows from the expression in (4.3) and the structure of the stopping times in (2.6) that the equalities

$$e^{r(\tau_i^* \wedge \varkappa_i^n)}\, V_i(S_{\tau_i^* \wedge \varkappa_i^n}) = V_i(s) + M^i_{\tau_i^* \wedge \varkappa_i^n} \tag{4.8}$$

are satisfied, for all $a_* < s < b'$ or $a' < s < b_*$, and any localising sequence $(\varkappa_i^n)_{n \in \mathbb{N}}$ of M^i, $i = 1, 2$. Observe that the form of the gain functions $G_i(s)$ together with the explicit expressions for the candidate value functions in (3.7)-(3.8), (3.17)-(3.18), and (3.29)-(3.30) yield that the conditions

$$E_s\Big[\sup_{t \ge 0} e^{r(\tau_i^* \wedge t)}\, V_i(S_{\tau_i^* \wedge t})\Big] < \infty \tag{4.9}$$

hold, for all $a_* < s < b'$ and $a' < s < b_*$, as well as the variables $e^{r\tau^*} V_i(S_{\tau_i^*})$ are bounded on the events $\{\tau_i^* = \infty\}$, $i = 1, 2$ (P_s-a.s.). Hence, taking into account the property in (4.9), we conclude from the expression in (4.8) that the processes $(M^i_{\tau_i^* \wedge t})_{t \ge 0}$, $i = 1, 2$, are uniformly integrable martingales. Therefore, taking the expectations in (4.8) and letting n go to infinity, we apply the Lebesgue dominated convergence theorem to obtain the equalities

$$E_s\big[e^{r\tau_i^*}\, G_i(S_{\tau_i^*})\big] = E_s\big[e^{r\tau_i^*}\, V_i(S_{\tau_i^*})\big] = V_i(s) \tag{4.10}$$

for all $a_* < s < b'$ and $a' < s < b_*$, and every $i = 1, 2$. The latter, together with the inequalities in (4.7), implies the fact that $V_i(s)$ coincide with the value functions $V_i^*(s)$, $i = 1, 2$, from (2.3).

□

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