

# Solving the dual Russian option problem by using change-of-measure arguments

Pavel V. Gapeev\*

We apply the change-of-measure arguments of Shepp and Shiryaev [38] to study the dual Russian option pricing problem proposed by Shepp and Shiryaev [39] as an optimal stopping problem for a one-dimensional diffusion process with reflection. We recall the solution to the associated free-boundary problem and give a solution to the resulting one-dimensional optimal stopping problem by using the martingale approach of Beibel and Lerche [6] and [7].

## 1 Introduction

The articles of Shepp and Shiryaev [37] and [38] initiated a large research area of pricing of derivative securities of American type with payoffs depending on the running values of the maxima of the underlying processes. The original Russian (put) option pricing problem of maximising the expected value of the discounted running maximum of a geometric Brownian motion was proposed and explicitly solved in [37] as an optimal stopping problem for a two-dimensional continuous Markov process. It was further observed by Shepp and Shiryaev [38] that the original Russian option problem can be reduced by means of the change-of-measure arguments to an optimal stopping problem for a one-dimensional diffusion process with reflection. Building on the optimal stopping analysis of [37] and [38], Duffie and Harrison [11] derived rational economic values for the Russian options and then extended their arbitrage arguments to more

---

\* London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

*Mathematics Subject Classification 2010:* Primary 60G40, 60G44, 60J65. Secondary 91B25, 60J60, 35R35.

*Key words and phrases:* Dual Russian option, optimal stopping problem, Brownian motion, diffusion process with reflection, first hitting time, free-boundary problem, martingale approach of Beibel and Lerche, change-of-variable formula with local time on surfaces.

*Date:* 19 Feb 2019

general perpetual American lookback options. Gerber et al. [17] and Mordecki and Moreira [28] obtained closed-form solutions to the perpetual Russian option problems for diffusions with negative exponential jumps. Pedersen [30] and Guo and Shepp [21] obtained closed-form solutions to the problems of pricing of more complicated perpetual American lookback options with payoffs depending on the running maxima processes of the underlying geometric Brownian motions. Shepp et al. [40] proposed a barrier version of the Russian option in the same model, where the decision about stopping should be taken before the price process reaches a certain positive level. Asmussen et al. [1] derived explicit expressions for the prices of Russian options in models with Lévy processes of both positive and negative jumps, by means of reducing the initial problems to the first passage time problems and solving the latter problems by the martingale stopping and Wiener-Hopf factorisation. Avram et al. [2] studied exit problems for spectrally negative Lévy processes and applied the results to solving optimal stopping problems associated with perpetual Russian and American put options. Peskir [32] presented a solution to the Russian option problem for a geometric Brownian motion with a finite time horizon (see also Duistermaat et al. [12] for a numeric algorithm of solving the corresponding free-boundary problem and Ekström [13] for a study of asymptotic behaviour of the optimal stopping boundary near the expiration). The problems of pricing of perpetual American lookback and other options with more complicated structure of payoffs depending on the running maxima of the underlying processes were studied in Gapeev [15], Baurdoux and Kyprianou [5], Guo and Zervos [22], Ott [29], Kyprianou and Ott [25], and Rodosthenous and Zervos [36] among others (see also Gapeev [14] and Kitapbayev [24] for the finite-time horizon American lookback options on the running maxima of geometric Brownian motions). Along with the article of Dubins et al. [10], the papers [37] and [38] also made a crucial contribution to the optimal stopping problems arising in the proofs of maximal inequalities for the continuous-time processes further developed in Graversen and Peskir [18]-[19] and Peskir [31] among others (see also Peskir and Shiryaev [34; Chapter V] for an extensive overview of the optimal stopping problems related to maximal inequalities).

Shepp and Shiryaev [39] proposed and explicitly solved the dual Russian (call) option pricing problem of (2.4) as an optimal stopping problem for a two-dimensional continuous Markov process  $(X, Y) = (X_t, Y_t)_{t \geq 0}$  defined in (2.1)–(2.3). The idea of writing this paper was to draw the attention of the readership to the article [39] in which an optimal stopping problem with positive exponential discounting rates was studied for one of the first times to the best of our knowledge. Other optimal stopping problems with positive exponential discounting rates were recently considered by Xia and Zhou [43], Battauz et al. [3]–[4], and De Donno et al. [9] among others. The introduction of positive exponential discounting rates into the optimal stopping problems implied the appearance of disconnected continuation regions or so-called

double continuation regions for the underlying processes. In this paper, we present an explicit solution to the optimal stopping problem of (2.4) by means of reducing it to the optimal stopping problem of (2.8) for a one-dimensional continuous Markov process  $Z = (Z_t)_{t \geq 0}$  solving the stochastic differential equation in (2.9) with the reflection term  $L = (L_t)_{t \geq 0}$  from (2.10)–(2.11). This paper can therefore be considered as a companion one to [38] in the sense that it solves the dual Russian option problem of [39] by means of the change-of-measure arguments. We also give a proof of the result by using the martingale approach suggested by Beibel and Lerche [6] and [7].

The paper is organised as follows. In Section 2, we introduce the setting and notation of the dual Russian option pricing optimal stopping problem of [39] and follow the change-of-measure arguments of [38] to reduce it to an optimal stopping problem for a one-dimensional diffusion process with reflection. In Section 3, we derive explicit solutions of the associated free-boundary problem for three different cases of relations between the parameters of the model. In Section 4, we verify that the solution of the free-boundary problem provides the solution of the initial optimal stopping problem of [39]. We also give a solution to the problem by means of the martingale approach of [6] and [7]. The main results of the paper are stated in Theorem 4.1.

## 2 Preliminaries

In this section, we recall the formulation of the dual Russian option problem proposed in [39] and the solution of the associated free-boundary problem.

### 2.1 The model

For a precise formulation of the problem, let us consider a probability space  $(\Omega, \mathcal{F}, P)$  with a standard Brownian motion  $B = (B_t)_{t \geq 0}$  and its natural filtration  $(\mathcal{F}_t)_{t \geq 0}$ . Let us define the process  $X = (X_t)_{t \geq 0}$  by:

$$X_t = x \exp \left( (r - \delta - \sigma^2/2) t + \sigma B_t \right) \quad (2.1)$$

which solves the stochastic differential equation:

$$dX_t = (r - \delta) X_t dt + \sigma X_t dB_t \quad (X_0 = x) \quad (2.2)$$

where  $x > 0$  is fixed, and  $r > 0$ ,  $\delta > 0$ , and  $\sigma > 0$  are some given constants. (Note that  $r - \delta$  in (2.1)–(2.2) corresponds to  $\mu$  in the notations of [39].) It is assumed that the process  $X$  describes the price of a risky asset on a financial market, where  $r$  is the riskless interest rate of a bank account,  $\delta$  is the dividend rate paid to the asset holders, and  $\sigma$  is the volatility rate. Let us now

define the associated with  $X$  *running minimum* process  $Y = (Y_t)_{t \geq 0}$  by:

$$Y_t = y \wedge \min_{0 \leq s \leq t} X_s \quad (2.3)$$

for an arbitrary  $0 < y \leq x$ . We study the optimal stopping problem with the value:

$$V_* = \inf_{\tau} E[e^{r\tau} Y_{\tau}] \quad (2.4)$$

where the infimum is taken over all stopping times  $\tau$  with respect to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . The problem of (2.4) was introduced and solved in [39] as an optimal stopping problem for a two-dimensional continuous (time-homogeneous strong) Markov process  $(X, Y) = (X_t, Y_t)_{t \geq 0}$ . In the present paper, we derive a solution to the problem of (2.4), by means of reducing it to an optimal stopping problem for a one-dimensional Markov process, and by applying the martingale approach from [6] and [7] (see also [26] and [16] among others).

In order to reduce the problem of (2.4) to a one-dimensional optimal stopping problem, let us follow the arguments of [38] and introduce the probability measure  $\tilde{P}$  defined by:

$$\left. \frac{d\tilde{P}}{dP} \right|_{\mathcal{F}_t} = \exp\left(\sigma B_t - (\sigma^2/2)t\right) \equiv e^{-(r-\delta)t} \frac{X_t}{x} \quad (2.5)$$

for all  $t \geq 0$ , so that  $\tilde{P}$  is locally equivalent to  $P$  on the filtration  $(\mathcal{F}_t)_{t \geq 0}$ , according to [35; Appendix, Theorem 6.1]. Then, by virtue of Girsanov's theorem (see, e.g. [35; Chapter VIII, Theorem 1.12]), we obtain that the process  $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$  defined by  $\tilde{B}_t = B_t - \sigma t$ , for all  $t \geq 0$ , is a standard Brownian motion under  $\tilde{P}$ . It thus follows from the expressions in (2.1)–(2.2) that the process  $X$  admits the representation:

$$X_t = x \exp\left((r - \delta + \sigma^2/2)t + \sigma \tilde{B}_t\right) \quad (2.6)$$

and solves the stochastic differential equation:

$$dX_t = (r - \delta + \sigma^2) X_t dt + \sigma X_t d\tilde{B}_t \quad (X_0 = x) \quad (2.7)$$

under  $\tilde{P}$ . It therefore follows that the value in (2.4) takes the form  $V_* = U_*/x$  with:

$$U_* = \inf_{\tau} \tilde{E}[e^{\lambda\tau} Z_{\tau}] \quad (2.8)$$

where the infimum is taken over the class of stopping times  $\tau$  with respect to  $(\mathcal{F}_t)_{t \geq 0}$ , and we set  $\lambda = 2r - \delta$ . Here, the process  $Z = (Z_t)_{t \geq 0}$  is defined by  $Z_t = Y_t/X_t$ , for all  $t \geq 0$ , and satisfies the stochastic differential equation:

$$dZ_t = -(r - \delta) Z_t dt - \sigma Z_t d\tilde{B}_t - dL_t \quad (Z_0 = z) \quad (2.9)$$

with  $z = (y/x) \wedge 1$ , where the process  $L = (L_t)_{t \geq 0}$  defined by:

$$L_t = - \int_0^t \frac{dY_s}{X_s} \quad (2.10)$$

is non-decreasing which increases when  $Z_t = 1$  and does not increase when  $Z_t < 1$ , for  $t \geq 0$ . Thus, we may conclude that the process  $L$  defined in (2.10) admits the representation:

$$L_t = - \int_0^t I(Z_s = 1) \frac{dY_s}{X_s} \quad (2.11)$$

for all  $t \geq 0$ , where  $I(\cdot)$  denotes the indicator function.

## 2.2 The optimal stopping problem

In order to compute the value of (2.8), let us therefore consider the optimal stopping problem:

$$U_*(z) = \inf_{\tau} \tilde{E}_z [e^{\lambda\tau} Z_{\tau}] \quad (2.12)$$

where the infimum is taken over stopping times  $\tau$  of the one-dimensional continuous Markov process  $Z$ , and we recall that  $\lambda = 2r - \delta$ . Here,  $\tilde{E}_z$  denotes the expectation with respect to the probability measure  $\tilde{P}$  under the assumption that the process  $Z$  starts at  $0 < z \leq 1$ . (Note that  $U_*(y/x)/x$  corresponds to  $V_*(x, y)$  in the notations of [39].) Moreover, it follows from the result of [39; Section 2, Lemma] that if  $0 < r \leq (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta < \sigma^2/2$  holds, then  $U_*(z) = 0$ , for all  $0 < z \leq 1$ . We further consider the case in which either  $r - \delta > \sigma^2/2$  or  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds and will seek for the the optimal stopping time in the problem of (2.12) in the form:

$$\tau_* = \inf\{t \geq 0 \mid Z_t \leq a_*\} \quad (2.13)$$

for some number  $0 < a_* < 1$  to be determined. (Note that  $a_*$  in (2.13) corresponds to  $1/\theta$  in the notations of [39].)

## 2.3 The free-boundary problem

It can be shown by means of standard arguments (see, e.g. [23; Chapter V, Section 5.1]) that the infinitesimal operator  $\mathbb{L}$  of the process  $Z$  acts on an arbitrary twice continuously differentiable locally bounded function  $F(z)$  according to the rule:

$$(\mathbb{L}F)(z) = -(r - \delta) z F'(z) + \frac{\sigma^2 z^2}{2} F''(z) \quad (2.14)$$

for all  $0 < z \leq 1$  (see also [8; Appendix 1, Section 2]). In order to find explicit expressions for the unknown value function  $U_*(z)$  from (2.12) and the unknown boundary  $a_*$  from (2.13), we

may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [41; Chapter III, Section 8] and [34; Chapter IV, Section 8]). We formulate the associated free-boundary problem:

$$(\mathbb{L}U)(z) = -\lambda U(z) \quad \text{for } a < z < 1 \quad (2.15)$$

$$U(a+) = a \quad (\text{instantaneous stopping}) \quad (2.16)$$

$$U'(a+) = 1 \quad (\text{smooth fit}) \quad (2.17)$$

$$U'(1-) = 0 \quad (\text{normal reflection}) \quad (2.18)$$

$$U(z) = z \quad \text{for } z < a \quad (2.19)$$

$$U(z) < z \quad \text{for } a < z < 1 \quad (2.20)$$

for some number  $0 < a < 1$  to be determined. Observe that the superharmonic characterisation of the value function (see, e.g. [41] and [34; Chapter IV, Section 9]) implies that  $U_*(z)$  is the smallest function satisfying (2.15)-(2.16) and (2.19)-(2.20) with the boundary  $a_*$ .

### 3 Solution to the free-boundary problem

We now look for functions which solve the free-boundary problem stated in (2.15)–(2.17). For this purpose, we consider three separate cases based on the different relations between the parameters of the model.

#### 3.1 The case $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$

Let us first assume that  $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds. Then, the general solution of the second-order ordinary differential equation in (2.15) has the form:

$$U(z) = C_1 z^{\eta_1} + C_2 z^{\eta_2} \quad (3.1)$$

where  $C_j$ ,  $j = 1, 2$ , are some arbitrary constants, and  $\eta_j$ ,  $j = 1, 2$ , are given by:

$$\eta_j = \frac{1}{2} + \frac{r - \delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} + \frac{r - \delta}{\sigma^2}\right)^2 - \frac{2\lambda}{\sigma^2}} \equiv \frac{1}{2} + \frac{r - \delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2 - \frac{2r}{\sigma^2}} \quad (3.2)$$

so that  $1 < \eta_2 < \eta_1$  in this case. (Note that  $\eta_j$ ,  $j = 1, 2$ , in (3.2) correspond to  $1 - \gamma_j$ ,  $j = 1, 2$ , in the notations of [39].) Hence, by applying the conditions from (2.16)–(2.18) to the function in (3.1), we get that the equalities:

$$C_1 a^{\eta_1} + C_2 a^{\eta_2} = a \quad (3.3)$$

$$C_1 \eta_1 a^{\eta_1} + C_2 \eta_2 a^{\eta_2} = a \quad (3.4)$$

$$C_1 \eta_1 + C_2 \eta_2 = 0 \quad (3.5)$$

should hold for some  $0 < a < 1$ . Thus, solving the system in (3.3)–(3.5), we obtain that the candidate value function has the form:

$$U(z; a_*) = a_* \left( \frac{\eta_2 - 1}{\eta_2 - \eta_1} \left( \frac{z}{a_*} \right)^{\eta_1} - \frac{\eta_1 - 1}{\eta_2 - \eta_1} \left( \frac{z}{a_*} \right)^{\eta_2} \right) \quad (3.6)$$

for  $a_* < z \leq 1$ , with

$$a_* = \left( \frac{\eta_1(\eta_2 - 1)}{\eta_2(\eta_1 - 1)} \right)^{1/(\eta_1 - \eta_2)}. \quad (3.7)$$

### 3.2 The case $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$ and $r - \delta > \sigma^2/2$

Let us now assume that  $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds. Then, the general solution of the ordinary differential equation in (2.15) has the form:

$$U(z) = C_1 z^\alpha \ln z + C_2 z^\alpha \quad (3.8)$$

where  $C_j$ ,  $j = 1, 2$ , are some arbitrary constants, and  $\alpha$  is given by:

$$\alpha = \frac{1}{2} + \frac{r - \delta}{\sigma^2} \quad (3.9)$$

so that  $\alpha > 1$  in this case. (Note that  $\alpha$  in (3.9) corresponds to  $1 - \gamma$  in the notations of [39].) Hence, by applying the conditions from (2.16)–(2.18) to the function in (3.8), we get that the equalities:

$$C_1 a^\alpha \ln a + C_2 a^\alpha = a \quad (3.10)$$

$$C_1 \alpha a^\alpha \ln a + C_1 a^\alpha + C_2 \alpha a^\alpha = a \quad (3.11)$$

$$C_1 + C_2 \alpha = 0 \quad (3.12)$$

should hold for some  $0 < a < 1$ . Thus, solving the system in (3.10)–(3.12), we obtain that the candidate value function has the form:

$$U(z; a_*) = a_* \left( (1 - \alpha) \left( \frac{z}{a_*} \right)^\alpha \ln \left( \frac{z}{a_*} \right) + \left( \frac{z}{a_*} \right)^\alpha \right) \quad (3.13)$$

for  $a_* < z \leq 1$ , with

$$a_* = \exp \left( \frac{1}{\alpha(1 - \alpha)} \right). \quad (3.14)$$

### 3.3 The case $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$

Let us finally assume that  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds. Then, the general solution of the ordinary differential equation in (2.15) has the form:

$$U(z) = C_1 z^\alpha \sin(\beta \ln z) + C_2 z^\alpha \cos(\beta \ln z) \quad (3.15)$$

where  $C_j$ ,  $j = 1, 2$ , are some arbitrary constants, while  $\alpha$  is given by (3.9) and  $\beta$  is defined as:

$$\beta = \sqrt{\frac{2\lambda}{\sigma^2} - \left(\frac{1}{2} + \frac{r - \delta}{\sigma^2}\right)^2} \equiv \sqrt{\frac{2r}{\sigma^2} - \left(\frac{1}{2} - \frac{r - \delta}{\sigma^2}\right)^2}. \quad (3.16)$$

Hence, by applying the conditions from (2.16)–(2.18) to the function in (3.15), we get that the equalities:

$$C_1 a^\alpha \sin(\beta \ln a) + C_2 a^\alpha \cos(\beta \ln a) = a \quad (3.17)$$

$$(C_1 \alpha - C_2 \beta) a^\alpha \sin(\beta \ln a) + (C_1 \beta + C_2 \alpha) a^\alpha \cos(\beta \ln a) = a \quad (3.18)$$

$$C_1 \beta + C_2 \alpha = 0 \quad (3.19)$$

should hold for some  $0 < a < 1$ . Thus, solving the system in (3.17)–(3.19), we obtain that the candidate value function has the form:

$$U(z; a_*) = a_* \left( \frac{1 - \alpha}{\beta} \left(\frac{z}{a_*}\right)^\alpha \sin\left(\beta \ln\left(\frac{z}{a_*}\right)\right) + \left(\frac{z}{a_*}\right)^\alpha \cos\left(\beta \ln\left(\frac{z}{a_*}\right)\right) \right) \quad (3.20)$$

for  $a_* < z \leq 1$ , with

$$a_* = \exp\left(\frac{1}{\beta} \arctan\left(\frac{\beta}{\alpha - \alpha^2 - \beta^2}\right)\right). \quad (3.21)$$

(Note that  $\arctan(\beta/(\alpha - \alpha^2 - \beta^2))$  in (3.21) corresponds to  $-\phi$  in the notations of [39].)

## 4 Main results and proofs

In this section, we show that the solution of the free-boundary problem from the previous section provides the solution of the initial optimal stopping problem of [39].

### 4.1 The verification theorem

Let us first verify that the solution of the free-boundary problem (2.14)–(2.20) coincides with the solution of the optimal stopping problem in (2.12). The following assertion was proved in [39] for the solution of the corresponding two-dimensional optimal stopping problem.

**Theorem 4.1** *Let the processes  $X$  and  $Y$  be given by (2.1)–(2.3), so that the process  $Z = Y/X$  admits the representation in (2.9). Then, the value function of the dual Russian optimal stopping problem in (2.12) has the form:*

$$U_*(z) = \begin{cases} U(z; a_*), & \text{if } a_* < z \leq 1 \\ z, & \text{if } 0 < z \leq a_* \end{cases} \quad (4.1)$$



and  $\tau_*$  from (2.13) is an optimal stopping time which is finite ( $\tilde{P}$ -a.s.), where we have:

(i) if  $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, then  $U(z; a_*)$  takes the expression of (3.6) and  $a_*$  is given by (3.7);

(ii) if  $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, then  $U(z; a_*)$  takes the expression of (3.13) and  $a_*$  is given by (3.14);

(iii) if  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds, then  $U(z; a_*)$  takes the expression of (3.20) and  $a_*$  is given by (3.21).

**Proof:** In order to verify the assertions stated above, let us show that the function defined in (4.1) coincides with the value function in (2.12), and that the stopping time  $\tau_*$  in (2.13) is optimal with the boundary  $a_*$  specified above. For this purpose, let us denote by  $U(z)$  the right-hand side of the expression in (4.1). Then, by means of straightforward calculations from the previous section, it is shown that the function  $U(z)$  solves the system of (2.15) with (2.19)–(2.20) and satisfies the conditions of (2.16)–(2.18). Hence, by applying the change-of-variable formula from [33] to the process  $e^{\lambda t}U(Z_t)$  (see also [34; Chapter II, Section 3.5] for a summary of the related results on the local time-space formula as well as further references), we obtain:

$$e^{\lambda t} U(Z_t) = U(z) + \int_0^t e^{\lambda s} (\mathbb{L}U + \lambda U)(Z_s) I(Z_s \neq a_*, Z_s \neq 1) ds + N_t \quad (4.2)$$

for all  $t \geq 0$ , where the process  $N = (N_t)_{t \geq 0}$  defined by:

$$N_t = - \int_0^t e^{\lambda s} U'(Z_s) \sigma Z_s d\tilde{B}_s \quad (4.3)$$

is a continuous local martingale with respect to the probability measure  $\tilde{P}_z$ .

By using straightforward calculations and the arguments from the previous section, it is verified that the inequality  $(\mathbb{L}V + \lambda V)(z) \equiv rz \geq 0$  holds for any  $0 < z < 1$  such that  $z \neq a_*$ . Moreover, it is shown by means of standard arguments that the inequality in (2.20) holds, which together with the conditions of (2.16)–(2.19) imply that  $U(z) \leq z$  holds, for all  $0 < z \leq 1$ . We also observe that the identity:

$$\int_0^t I(Z_s = 1) ds = 0 \quad (\tilde{P}_z - \text{a.s.}) \quad (4.4)$$

holds, for all  $t \geq 0$ . This property follows from the fact that the couple of processes  $(X, Y)$  has a density with respect to the Lebesgue measure, so that  $\tilde{P}_z(Z_t = 1) = 0$ , for all  $t \geq 0$ , and by virtue of Fubini's theorem, we get that the equality:

$$\tilde{E}_z \int_0^\infty I(Z_s = 1) ds = \int_0^\infty \tilde{E}_z [I(Z_s = 1)] ds = \int_0^\infty \tilde{P}_z(Z_s = 1) ds = 0 \quad (4.5)$$

holds (see also [42; Chapter VIII, Section 2] for the proof related to the standard Russian put option). Thus, taking into account the fact that the time spent by the process  $Z$  at the point  $a_*$  is of the Lebesgue measure zero (see, e.g. [8; Chapter II, Section 1]), we may conclude that the indicator in (4.2) can be set to one. Hence, the expression in (4.2) yields that the inequalities:

$$e^{\lambda\tau} Z_\tau \geq e^{\lambda\tau} U(Z_\tau) \geq U(z) + N_\tau \quad (4.6)$$

hold, for any stopping time  $\tau$  of the process  $Z$  started at  $0 < z \leq 1$ .

Let  $(\tau_k)_{k \in \mathbb{N}}$  be the localising sequence of stopping times for the process  $N$  from (4.3) such that  $\tau_k = \inf\{t \geq 0 \mid |N_t| \geq k\}$ , for each  $k \in \mathbb{N}$ . It therefore follows from the expression in (4.2) that the inequalities:

$$e^{\lambda(\tau \wedge \tau_k)} Z_{\tau \wedge \tau_k} \geq e^{\lambda(\tau \wedge \tau_k)} U(Z_{\tau \wedge \tau_k}) \geq U(z) + N_{\tau \wedge \tau_k} \quad (4.7)$$

hold, for any stopping time  $\tau$  of the process  $Z$  and each  $k \in \mathbb{N}$  fixed. Taking the expectation with respect to  $\tilde{P}_z$  in (4.6), by means of Doob's optional sampling theorem (see, e.g. [23; Chapter I, Theorem 3.22]), we get that the inequalities:

$$\tilde{E}_z[e^{\lambda(\tau \wedge \tau_k)} Z_{\tau \wedge \tau_k}] \geq \tilde{E}_z[e^{\lambda(\tau \wedge \tau_k)} U(Z_{\tau \wedge \tau_k})] \geq U(z) + \tilde{E}_z[N_{\tau \wedge \tau_k}] = U(z) \quad (4.8)$$

hold, for all  $0 < z \leq 1$ . Hence, letting  $k$  go to infinity and using Fatou's lemma, we obtain that the inequalities:

$$\tilde{E}_z[e^{\lambda\tau} Z_\tau] \geq \tilde{E}_z[e^{\lambda\tau} U(Z_\tau)] \geq U(z) \quad (4.9)$$

are satisfied, for any stopping time  $\tau$  and all  $0 < z \leq 1$ . By virtue of the structure of the stopping time in (2.13), it is readily seen that the equalities in (4.9) hold with  $\tau_*$  instead of  $\tau$  when  $0 < z \leq a_*$ .

Let us now show that the stopping time  $\tau_*$  in (2.13) is finite ( $\tilde{P}$ -a.s.). For this purpose, we follow the schema of arguments of [6] and [42; Chapter VIII, Section 2]. We first observe that the expression:

$$\ln Z_t = \ln z \wedge \min_{0 \leq s \leq t} \left( \left( r - \delta + \frac{\sigma^2}{2} \right) (s - t) + \sigma (\tilde{B}_s - \tilde{B}_t) \right) \quad (4.10)$$

holds, for all  $t \geq 0$ . Then, we consider a sequence of first hitting times  $(\zeta_k)_{k \in \mathbb{N}}$  such that  $\zeta_k = \inf\{t \geq \zeta_{k-1} + 1 \mid \ln Z_t = 0\}$ , for  $k \in \mathbb{N}$ , and  $\zeta_0 = 0$ . In this case, we see that:

$$\left\{ \inf_{t \geq 0} \ln Z_t \leq \ln a_* \right\} = \bigcup_{k \in \mathbb{N}} \left\{ \inf_{\zeta_{k-1} \leq t < \zeta_k} \ln Z_t \leq \ln a_* \right\} \quad (4.11)$$

holds, where we have  $\ln a_* < 0$ . Note that the events on the right-hand side of (4.11) are independent for different  $k \in \mathbb{N}$ , and their probabilities are equal and strictly positive. Hence, by virtue of the Borel-Cantelli lemma, we may conclude that:

$$\tilde{P}_z \left( \inf_{t \geq 0} \ln Z_t \leq \ln a_* \right) = 1 \quad (4.12)$$

so that  $\tilde{P}_z(\tau_* < \infty) = 1$ , for any  $0 < z \leq 1$  fixed.

It remains us to show that the equalities are attained in (4.9) when  $\tau_*$  replaces  $\tau$ , for  $a_* < z \leq 1$ . By virtue of the fact that the function  $U(z; a_*)$  and the boundary  $a_*$  satisfy the conditions in (2.15) and (2.16), it follows from the expression in (4.2) and the structure of the stopping time in (2.13) that the equality:

$$e^{\lambda(\tau_* \wedge \tau_k)} U(Z_{\tau_* \wedge \tau_k}; a_*) = U(z) + N_{\tau_* \wedge \tau_k} \quad (4.13)$$

holds, for all  $a_* < z \leq 1$  and each  $k \in \mathbb{N}$ . Observe that the explicit form of the candidate value function in (3.6), (3.13), and (3.20) yields that the condition:

$$\tilde{E}_z \left[ \sup_{t \geq 0} e^{\lambda(\tau_* \wedge t)} U(Z_{\tau_* \wedge t}; a_*) \right] < \infty \quad (4.14)$$

holds, for all  $a_* < z \leq 1$ . Hence, taking into account the property in (4.14) as well as the fact that  $\tilde{P}_z(\tau_* < \infty) = 1$ , we conclude from the expression in (4.13) that the process  $(N_{\tau_* \wedge t})_{t \geq 0}$  is a uniformly integrable martingale. Therefore, taking the expectation in (4.13) and letting  $n$  go to infinity, we apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$\tilde{E}_z [e^{\lambda \tau_*} Z_{\tau_*}] = \tilde{E}_z [e^{\lambda \tau_*} U(Z_{\tau_*}; a_*)] = U(z) \quad (4.15)$$

for all  $a_* < z \leq 1$ . The latter, together with the inequalities in (4.9), implies the fact that  $U(z)$  coincides with the value function  $U_*(z)$  from (2.12).  $\square$

## 4.2 The martingale approach

Let us finally apply the martingale approach of [6] and [7] for solving the optimal stopping problem of (2.12). For this purpose, we first observe from the structure of the processes  $(X, Y)$  in (2.1)–(2.3) and of the reward in the optimal stopping problem of (2.12) that  $U_*(z) = z \tilde{U}_*(1) \equiv z \tilde{U}_*$  holds for the value function with

$$\tilde{U}_* = \inf_{\tau} \tilde{E}_1 [e^{\lambda \tau} Z_{\tau} I(\tau < \infty)] \equiv \tilde{E}_1 [e^{\lambda \tau_*} Z_{\tau_*} I(\tau_* < \infty)] \quad (4.16)$$

where the infimum is taken over all stopping times  $\tau$  of the process  $Z$ , and we have  $\tilde{P}_1(\tau_* < \infty) = 1$ . We now search for a function  $H(z)$  such that the process  $M = (M_t)_{t \geq 0}$  defined by:

$$M_t = e^{\lambda t} Z_t H(Z_t) \quad (4.17)$$

and started at  $M_0 = 1$  is a continuous local martingale under the probability measure  $\tilde{P}$ . In this case, by applying Itô's formula (see, e.g. [27; Theorem 4.4]) to the process  $e^{\lambda t} Z_t H(Z_t)$ , we obtain that the function  $H(z)$  should solve the second-order ordinary differential equation:

$$\frac{\sigma^2 z^2}{2} H''(z) - (r - \delta - \sigma^2) z H'(z) = (r - \delta - \lambda) H(z) \quad (4.18)$$

as well as satisfy the boundary conditions:

$$H(1-) = 1 \quad \text{and} \quad H'(1-) = -1. \quad (4.19)$$

It is shown by means of straightforward computations, similar to the ones derived in the previous section, that the function  $H(z)$  admits the representation:

$$H(z) = \frac{\eta_2}{\eta_2 - \eta_1} z^{\eta_1 - 1} - \frac{\eta_1}{\eta_2 - \eta_1} z^{\eta_2 - 1} \quad (4.20)$$

when  $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or

$$H(z) = -\alpha z^{\alpha-1} \ln z + z^{\alpha-1} \quad (4.21)$$

when  $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or

$$H(z) = -\frac{\alpha}{\beta} z^{\alpha-1} \sin(\beta \ln z) + z^{\alpha-1} \cos(\beta \ln z) \quad (4.22)$$

when  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds, respectively. By means of straightforward computations, it can be deduced from the expressions in (4.20), or (4.21), or (4.22), that the first- and second-order derivatives  $H'(z)$  and  $H''(z)$  of the function  $H(z)$  solving the equation in (4.18) and satisfying the conditions of (4.19) take the form:

$$H'(z) = \frac{\eta_2(\eta_1 - 1)}{\eta_2 - \eta_1} z^{\eta_1 - 2} - \frac{\eta_1(\eta_2 - 1)}{\eta_2 - \eta_1} z^{\eta_2 - 2} \quad (4.23)$$

and

$$H''(z) = \frac{\eta_2(\eta_1 - 1)(\eta_1 - 2)}{\eta_2 - \eta_1} z^{\eta_1 - 3} - \frac{\eta_1(\eta_2 - 1)(\eta_2 - 2)}{\eta_2 - \eta_1} z^{\eta_2 - 3} \quad (4.24)$$

when  $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or

$$H'(z) = -\alpha(\alpha - 1) z^{\alpha-2} \ln z - z^{\alpha-2} \quad (4.25)$$

and

$$H''(z) = -\alpha(\alpha - 1)(\alpha - 2) z^{\alpha-3} \ln z - (\alpha^2 - 2) z^{\alpha-3} \quad (4.26)$$

when  $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or

$$H'(z) = -\frac{\alpha^2 - \alpha + \beta^2}{\beta} z^{\alpha-2} \sin(\beta \ln z) - z^{\alpha-2} \cos(\beta \ln z) \quad (4.27)$$

and

$$H''(z) = -\frac{(\alpha^2 - \alpha + \beta^2)(\alpha - 2) - \beta^2}{\beta} z^{\alpha-3} \sin(\beta \ln z) - (\alpha^2 + \beta^2 - 2) z^{\alpha-3} \cos(\beta \ln z) \quad (4.28)$$

when  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds, respectively. It is shown by verifying the equation  $H'(a_*) = 0$  and the inequality  $H''(z) < 0$ , for all  $z \in (0, 1)$ , for the functions in (4.23)-(4.24), or (4.25)-(4.26), or (4.27)-(4.28), that the function  $H(z)$  in (4.20), or (4.21), or (4.22), attains its maximum at the point  $a_*$  given by (3.7), or (3.14), or (3.21), respectively (see also [42; Chapter VIII, Section 2] for the proof related to the standard Russian put option). Hence, it follows from the structure of the expressions in (4.20), or (4.21), or (4.22), that the process  $M$  defined in (4.17) is a continuous local martingale bounded above, and thus, it is a submartingale. Therefore, we obtain that the inequalities:

$$\begin{aligned} \tilde{E}_1[e^{\lambda\tau} Z_\tau I(\tau < \infty)] &= \tilde{E}_1[(1/H(Z_\tau)) M_\tau I(\tau < \infty)] \\ &\geq (1/H(a_*)) \tilde{E}_1[M_\tau I(\tau < \infty)] \geq 1/H(a_*) \end{aligned} \quad (4.29)$$

hold, for any finite stopping time  $\tau$  of the process  $Z$ . By virtue of the fact that  $\tilde{P}_1(\tau_* < \infty) = 1$ , we may therefore summarise the arguments above to conclude that the expressions:

$$\tilde{U}_* = \tilde{E}_1[(1/H(Z_{\tau_*})) M_{\tau_*} I(\tau_* < \infty)] = (1/H(a_*)) \tilde{E}_1[M_{\tau_*} I(\tau_* < \infty)] = 1/H(a_*) \quad (4.30)$$

hold and formulate the following assertion.

**Corollary 4.2** *Let the assumptions of Theorem 4.1 above hold. Then, the value function of the dual Russian optimal stopping problem in (2.12) admits the representation  $U_*(z) = z\tilde{U}_*(1) \equiv z\tilde{U}_*$  with  $\tilde{U}_*$  being of the form of (4.30), where  $H(z)$  and  $a_*$  take the expressions of (4.20) and (3.7) when  $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or (4.21) and (3.14) when  $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or (4.22) and (3.21) when  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds, respectively.*

**Remark 4.3** *Note that it can also be deduced from the arguments of the previous section that the function  $zH(z)$  should solve the second-order ordinary differential equation in (2.15), so that the function  $H(z)$  from (4.20), or (4.21), or (4.22), equivalently admits the representation:*

$$H(z) = \frac{U(z; a_*)}{zU(1; a_*)} \quad (4.31)$$

for all  $0 < z < 1$ . Here, the function  $U(z; a_*)$  is the solution of the equation in (2.15) with the boundary conditions in (2.16)–(2.17) which takes the expressions of (3.6) when  $0 < r < (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or (3.13) when  $r = (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  and  $r - \delta > \sigma^2/2$  holds, or (3.20) when  $r > (r - \delta - \sigma^2/2)^2/(2\sigma^2)$  holds, respectively.

**Acknowledgments.** The author is grateful to an anonymous Referee for their interesting suggestions which allowed to improve the presentation of the paper. The paper was initiated

when the author was visiting Albert-Ludwigs-Universität Freiburg im Breisgau in December 2014. The author is grateful to Hans Rudolf Lerche for fruitful discussions. The warm hospitality at the Abteilung für Mathematische Stochastik is gratefully acknowledged.

## References

- [1] ASMUSSEN, S., AVRAM, F. and PISTORIUS, M. (2003). Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and Applications* **109** (79–111).
- [2] AVRAM, F., KYPRIANOU, A. E. and PISTORIUS, M. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Annals of Applied Probability* **14**(1) (215–238).
- [3] BATTAUZ, A., DE DONNO, M. and SBUELZ, A. (2012). Real options with a double continuation region. *Quantitative Finance* **12**(3) (465–475).
- [4] BATTAUZ, A., DE DONNO, M. and SBUELZ, A. (2014). Real options and American derivatives: the double continuation region. *Management Science* **61**(5) (1094–1107).
- [5] BAURDOUX, E. J. and KYPRIANOU, A. E. (2009). The Shepp-Shiryaev stochastic game driven by a spectrally negative Lévy process. *Theory of Probability and its Applications* **53** (481–499).
- [6] BEIBEL, M. and LERCHE, H. R. (1997). A new look at optimal stopping problems related to mathematical finance. *Statistica Sinica* **7** (93–108).
- [7] BEIBEL, M. and LERCHE, H. R. (2000). A note on optimal stopping of regular diffusions under random discounting. *Theory of Probability and Its Applications* **45** (657–669).
- [8] BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian Motion*. (Second Edition) Birkhäuser, Basel.
- [9] DE DONNO, M., PALMOWSKI, Z. and TUMILEWICZ, J. (2018). Double continuation regions for American and swing options with negative discount rate in Lévy models. *Preprint* (24 pp).
- [10] DUBINS, L., SHEPP, L. A. and SHIRYAEV, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes. *Theory of Probability and Applications* **38**(2) (226–261).

- [11] DUFFIE, J. D. and HARRISON, J. M. (1993). Arbitrage pricing of Russian options and perpetual lookback options. *Annals of Applied Probability* **3**(3) (641–651).
- [12] DUISTERMAAT, J. J., KYPRIANOU, A. E. and VAN SCHAIK, K. (2005). Finite expiry Russian options. *Stochastic Processes and Applications* **115**(4) (609–638).
- [13] EKSTRÖM, E. (2004). Russian options with a finite time horizon. *Journal of Applied Probability* **41**(2) (313–326).
- [14] GAPEEV, P. V. (2006). Discounted optimal stopping for maxima in diffusion models with finite horizon. *Electronic Journal of Probability* **11** (1031–1048).
- [15] GAPEEV, P. V. (2007). Discounted optimal stopping for maxima of some jump-diffusion processes. *Journal of Applied Probability* **44** (713–731).
- [16] GAPEEV, P. V. and LERCHE, H. R. (2011). On the structure of discounted optimal stopping problems for one-dimensional diffusions. *Stochastics: An International Journal of Probability and Stochastic Processes* **83** (537–554).
- [17] GERBER, H. U., MICHAUD, F. and SHIU, E. S. W. (1995). Pricing Russian options with the compound Poisson process. *Insurance: Mathematics and Economics* **17**(1) (79–79).
- [18] GRAVERSEN, S. E. and PESKIR, G. (1998). Optimal stopping and maximal inequalities for geometric Brownian motion. *Journal of Applied Probability* **35**(4) (856–872).
- [19] GRAVERSEN, S. E. and PESKIR, G. (1998). Optimal stopping and maximal inequalities for linear diffusions. *Journal of Theoretical Probability* **11** (259–277).
- [20] GUO, X. (2001). An explicit solution to an optimal stopping problem with regime switching. *Journal of Applied Probability* **38** (464–481).
- [21] GUO, X. and SHEPP, L. A. (2001). Some optimal stopping problems with nontrivial boundaries for pricing exotic options. *Journal of Applied Probability* **38**(3) (647–658).
- [22] GUO, X. and ZERVOS, M. (2010).  $\pi$  options. *Stochastic Processes and Applications* **120**(7) (1033–1059).
- [23] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*. (Second Edition) Springer, New York.
- [24] KITAPBAYEV, Y. (2014). On the lookback option with fixed strike. *Stochastics: An International Journal of Probability and Stochastic Processes* **86** (510–526).

- [25] KYPRIANOU, A. E. and OTT, C. (2014). A capped optimal stopping problem for the maximum process. *Acta Applicandae Mathematicae* **129** (147–174).
- [26] LERCHE, H. R. and URUSOV, M. (2007). Optimal stopping via measure transformation: The Beibel–Lerche approach. *Stochastics: An International Journal of Probability and Stochastic Processes* **79** (275–291).
- [27] LIPTSER, R. S. and SHIRYAEV, A. N. (2001). *Statistics of Random Processes I.* (Second Edition, First Edition 1977) Springer, Berlin.
- [28] MORDECKI, E. and MOREIRA, W. (2001). Russian options for a diffusion with negative jumps. *Publicaciones Matemáticas del Uruguay* **9** (37–51).
- [29] OTT, C. (2013). Optimal stopping problems for the maximum process with upper and lower caps. *Annals of Applied Probability* **23** (2327–2356).
- [30] PEDERSEN, J. L. (2000). Discounted optimal stopping problems for the maximum process. *Journal of Applied Probability* **37**(4) (972–983).
- [31] PESKIR, G. (1998). Optimal stopping of the maximum process: The maximality principle. *Annals of Probability* **26**(4) (1614–1640).
- [32] PESKIR, G. (2005). The Russian option: finite horizon. *Finance and Stochastics* **9** (251–267).
- [33] PESKIR, G. (2007). A change-of-variable formula with local time on surfaces. *Séminaire de Probabilité XL. Lecture Notes in Mathematics* **1899**, Springer (69–96).
- [34] PESKIR, G. and SHIRYAEV, A. N. (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser, Basel.
- [35] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion.* (Third Edition) Springer, Berlin.
- [36] RODOSTHENOUS, N. and ZERVOS, M. (2017). Watermark options. *Finance and Stochastics* **21**(1) (157–186).
- [37] SHEPP, L. A. and SHIRYAEV, A. N. (1993). The Russian option: Reduced regret. *Annals of Applied Probability* **3** (631–640).
- [38] SHEPP, L. A. and SHIRYAEV, A. N. (1994). A new look at the pricing of Russian options. *Theory of Probability and Its Applications* **39** (103–119).



- [39] SHEPP, L. A. and SHIRYAEV, A. N. (1996). A dual Russian option for selling short. In *Probability Theory and Mathematical Statistics: Lectures presented at the semester held in St. Peterburg, Russia, March 2 April 23, 1993*. Ibragimov, I. A. et al. eds. Gordon and Breach, Amsterdam (209-218).
- [40] SHEPP, L. A., SHIRYAEV, A. N. and SULEM, A. (2002). A barrier version of the Russian option. *Advances in Finance and Stochastics*. Essays in Honour of Dieter Sondermann. Sandmann, K. and Schönbucher, P. eds. Springer, Berlin (271–284).
- [41] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules*. Springer, Berlin.
- [42] SHIRYAEV, A. N. (1999). *Essentials of Stochastic Finance*. World Scientific, Singapore.
- [43] XIA, J. and ZHOU, X. (2007). Stock loans. *Mathematical Finance* **17** (307–317).